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On an Approximate Solution for the Bending of a Beam of Rectangular Cross-Section under any System of Load, with Special Reference to Points of Concentrated or Discontinuous Loading

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IV. *On an Approximate Solution for the Bending of a Beam of Rectangular Cross-Section under any System of Load, with Special Reference to Points of Concentrated or Discontinuous Loading.*

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PART I.

Establishment and General Solution of the Equations of the Problem Discussed.

§ 1. *General Sketch of the Problem proposed.*

The consideration of the stresses and strains which occur in a rectangular parallelopiped of elastic material subjected to given surface forces over its six faces leads to one of the most general, as it is one of the oldest, problems in the Theory of Elasticity. Lamé, in his 'Leçons sur l'Élasticité des Corps solides,' published in 1852, describes it as "le plus difficile peut-être de la théorie mathématique de l'élasticité." In spite of repeated attempts, however, the problem remains still unsolved.

In its complete form it may be stated as follows:—

Let the origin be taken at the centre of the parallelopiped and the axes $0x$, $0y$, $0z$ parallel to its edges. Let the lengths of these edges be $2a$, $2b$, $2c$. Let u, v, w denote the displacements of any point (x, y, z) parallel to the three axes, and, following the notation of Todhunter and Pearson's 'History of Elasticity,' let $\widehat{st}$ denote the stress, parallel to s, across an elementary area perpendicular to t, then we have the six stresses

$$\left.\begin{aligned}\widehat{xx} &= \lambda\delta + 2\mu\frac{du}{dx} & \widehat{yz} &= \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\\ \widehat{yy} &= \lambda\delta + 2\mu\frac{dv}{dy} & \widehat{zx} &= \mu\left(\frac{dw}{dx} + \frac{du}{dz}\right)\\ \widehat{zz} &= \lambda\delta + 2\mu\frac{dw}{dz} & \widehat{xy} &= \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, and λ, μ are the elastic constants of Lamé.

Also u, v, w must satisfy, inside the material, the following differential equations,

$$\left.\begin{aligned}(\lambda + \mu)\frac{d\delta}{dx} + \mu\nabla^2 u &= 0\\ (\lambda + \mu)\frac{d\delta}{dy} + \mu\nabla^2 v &= 0\\ (\lambda + \mu)\frac{d\delta}{dz} + \mu\nabla^2 w &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $\nabla^2 \equiv \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, there being no body force acting on the matter inside the block. It is required to find the values of u, v, w at each point, subject to the condition that the stress across the outer faces $x = \pm a$, $y = \pm b$, $z = \pm c$ shall be arbitrarily given at each point—regard being had, of course, to the conditions of rigid equilibrium of the block.

Since LAMÉ'S time the problem has been attacked by a large number of mathematicians, among them DE SAINT-VENANT, CLEBSCH, BOUSSINESQ, and more recently M. MATHIEU, M. RIBIÈRE and Mr. J. H. MICHELL. Although they have not been able so far to obtain the solution of the problem as stated quite generally above, they have nevertheless made great progress with various particular cases, more especially those in which some of the dimensions of the block are large compared with the rest.

Fuller references to their work and to the results obtained by them are given in the historical summary at the end of this paper.

§ 2. *Object of the Investigation.*

The object of the present investigation is to obtain the solution for the rectangular parallelopiped under an arbitrary system of surface loading in two cases, when the problem reduces to one of two dimensions, namely :—

(A) When two of the faces $z = \pm c$ of the bar are constrained to remain plane and the stress applied to the other faces is independent of z. In this case $w = 0$, u and v are functions of x and y only. If the breadth $2c$ of the beam be sufficiently large, we may relinquish the constraint along the sides altogether, and we have thus the case of a thick plate bent in a plane perpendicular to its own plane. When the plate is made indefinitely thick we have two-dimensional strain in an infinite elastic solid with a plane boundary.

(B) When we make the assumption that $\widehat{xz}$ and $\widehat{yz}$ vanish at the boundaries $z = \pm c$, while $\widehat{zz}$ is actually zero throughout. That this will be very near the truth if c is very small is quite evident, so that in any case this condition will hold for a flat beam or girder whose height is large compared with its breadth.*

But it seems not improbable that it may continue to hold approximately up to a fairly large value of c; we may remember that DE SAINT-VENANT, in his solution for flexure, assumes both $\widehat{zz}$ and $\widehat{yy}$ to be zero, in the case where his beam is unstressed except at the ends, and his solution is sufficient to satisfy all conditions. Obviously vertical pressures and tensions across the faces $y = \pm b$ must introduce important stresses $\widehat{yy}$, so that that part of DE SAINT-VENANT'S hypothesis, in the generalised problem, must go. Still it appears reasonable to suppose, on the whole, that, even for a beam where c and b are of the same order, we may, as a first approximation, retain the hypothesis $\widehat{zz} = 0$. Of course, eventually, as c increases a stress $\widehat{zz}$ must appear until when c is very large we reach the limiting case of problem (A) when this stress is sufficient to ensure the vanishing of the displacement w.

If, however, c be not too large, so that we can suppose $\widehat{zz}$ sensibly zero throughout,

* September 13, 1902. I have, since writing the above, verified that a solution for rectangular beams does exist, which fulfils *rigidly* these conditions. It is, in fact, identical with part of CLEBSCH'S solution for a thick plate.

then the *mean* values U, V taken across the breadth of the beam of the displacements u, v in the plane xy are found to satisfy two differential equations of the same form as the equations of elasticity when the displacements are independent of z and $w = 0$, with this change, that the elastic constant λ is replaced by another constant λ'. The *mean stresses* in the plane of xy are found by differentiation from U and V by similar formulæ to those giving $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ in terms of u, v for two-dimensional strain.

Now the distribution of such mean stresses inside the beam is independent of the ratio $\lambda' : \mu$. This has been shown by Mr. J. H. MICHELL ('London Mathematical Society's Proceedings,' vol. 31, pp. 100–124). It had been previously pointed out by STOKES ('Phil. Mag.,' Ser. V., vol. 32, p. 503). The equations being of the same form in problems (A) and (B), there follows this curious result, that the distribution of *stress* inside the beam, consequent upon a given distribution of stress upon the upper and lower faces (this latter distribution being uniform with regard to the breadth of the beam) is the same when this breadth is very small and when it is very large.

§ 3. *Establishment of the Equations.*

The centre of the rectangular beam being the origin, let its axis, which is supposed horizontal, be taken as axis of x. The axis of y will be vertical and the axis of z horizontal. The bounding surfaces of the beam are $x = \pm a$, $y = \pm b$, $z = \pm c$.

Using the notation explained in § 1, equations (2) may be written

$$\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz} = 0 \quad \ldots \ldots \ldots \ldots \quad (3),$$

$$\frac{d\widehat{xy}}{dx} + \frac{d\widehat{yy}}{dy} + \frac{d\widehat{yz}}{dz} = 0 \quad \ldots \ldots \ldots \ldots \quad (4),$$

$$\frac{d\widehat{xz}}{dx} + \frac{d\widehat{yz}}{dy} + \frac{d\widehat{zz}}{dz} = 0 \quad \ldots \ldots \ldots \ldots \quad (5).$$

Integrate equations (3) and (4) with regard to z from $-c$ to $+c$. Then, noting that $(\widehat{xz})_{z=\pm c}$, $(\widehat{yz})_{z=\pm c}$ are both zero, owing to the surface conditions at the side of the beam, and also that integration with regard to z and differentiation with regard to x and y are independent, we find

$$\frac{d}{dx}\left[\int_{-c}^{+c} \widehat{xx}\, dz\right] + \frac{d}{dy}\left[\int_{-c}^{+c} \widehat{xy}\, dz\right] = 0,$$

$$\frac{d}{dx}\left[\int_{-c}^{+c} \widehat{xy}\, dz\right] + \frac{d}{dy}\left[\int_{-c}^{+c} \widehat{yy}\, dz\right] = 0.$$

Now if we write $\int_{-c}^{+c} \widehat{xx}\, dz = 2c\text{P}$, $\int_{-c}^{+c} \widehat{yy}\, dz = 2c\text{Q}$, $\int_{-c}^{+c} \widehat{xy}\, dz = 2c\text{S}$, then P, Q, S are the mean values of the two tractions and of the shear in the plane xy—taken, for any values of x, y, across the breadth of the beam. These will in future be referred to as the mean stresses and often, for shortness, as the stresses.

We obtain, therefore, the equations

$$\frac{dP}{dx} + \frac{dS}{dy} = 0 \quad . \quad . \quad . \quad . \quad (6), \qquad \frac{dS}{dx} + \frac{dQ}{dy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now consider equations (1), namely,

$$\widehat{xx} = \lambda\left(\frac{du}{dx} + \frac{dv}{dy}\right) + 2\mu\frac{du}{dx} + \lambda\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$\widehat{yy} = \lambda\left(\frac{du}{dx} + \frac{dv}{dy}\right) + 2\mu\frac{dv}{dy} + \lambda\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\widehat{zz} = \lambda\left(\frac{du}{dx} + \frac{dv}{dy}\right) + (\lambda + 2\mu)\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$$\widehat{xy} = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

If we integrate (8), (9), and (11) with regard to z from $-c$ to $+c$, we have

$$P = \lambda\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dU}{dx} + \lambda\left(\frac{w_{+c} - w_{-c}}{2c}\right) \quad . \quad . \quad . \quad . \quad . \quad (12),$$

$$Q = \lambda\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dV}{dy} + \lambda\left(\frac{w_{+c} - w_{-c}}{2c}\right) \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$S = \mu\left(\frac{dU}{dy} + \frac{dV}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where $U = \frac{1}{2c}\int_{-c}^{+c} u\, dz$, $V = \frac{1}{2c}\int_{-c}^{+c} v\, dz$ are the *mean* displacements in the plane of xy taken across the breadth of the beam for any point (x, y). They will be referred to as the mean displacements. Besides these there is a variable $(w_{+c} - w_{-c})/2c$ which has to be eliminated somehow.

One way of doing this is by integrating (10) in the same way. We obtain

$$\frac{1}{2c}\int_{-c}^{+c} \widehat{zz}\, dz = \lambda\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + (\lambda + 2\mu)\left(\frac{w_{+c} - w_{-c}}{2c}\right).$$

Now if, as explained in the last section, $\widehat{zz}$ may be treated as small, so that its mean value across the breadth of the beam may be neglected, we have

$$\frac{w_{+c} - w_{-c}}{2c} = -\frac{\lambda}{\lambda + 2\mu}\left(\frac{dU}{dx} + \frac{dV}{dy}\right).$$

Substituting for $(w_{+c} - w_{-c})/2c$, the equations for P and Q become

$$P = \lambda'\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dU}{dx} \quad . \quad . \quad . \quad (15),$$

$$Q = \lambda'\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dV}{dy} \quad . \quad . \quad . \quad (16),$$

where $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$. Putting these into (6) and (7), we have

$$(\lambda' + \mu)\frac{d}{dx}\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + \mu\nabla^2 U = 0 \quad . \quad . \quad . \quad (17),$$

$$(\lambda' + \mu)\frac{d}{dy}\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + \mu\nabla^2 V = 0 \quad . \quad . \quad . \quad (18).$$

(15), (16), (17), and (18) are precisely of the same form as the stress-strain relations and the body equations of equilibrium for two-dimensional elastic strain, with the exception that λ' is written for λ. They will in fact be found to be identical with the equations satisfied by the displacements of an elastic plate under thrust in its own plane, as obviously they should be, since, when the beam is made indefinitely thin, the mean displacements U, V coincide with the actual displacements u, v.

§ 4. *General Solution of the Equations in Arbitrary Functions.*

If we write $\frac{dU}{dx} + \frac{dV}{dy} = \delta$, $x + iy = \xi$, $x - iy = \eta$, where $i = \sqrt{-1}$, so that $\frac{d}{dx} + i\frac{d}{dy} = 2\frac{d}{d\eta}$ and $\frac{d}{dx} - i\frac{d}{dy} = 2\frac{d}{d\xi}$, multiply (18) by i and add to (17), we find

$$2(\lambda' + \mu)\frac{d\delta}{d\eta} + \mu\nabla^2(U + iV) = 0.$$

Multiply (18) by i and subtract from (17)

$$2(\lambda' + \mu)\frac{d\delta}{d\xi} + \mu\nabla^2(U - iV) = 0.$$

But $\nabla^2 = 4\frac{d^2}{d\xi\, d\eta}$, and if $T = U + iV$, $W = U - iV$, then

$$\delta = \frac{dU}{dx} + \frac{dV}{dy} = \frac{d}{d\xi}(U + iV) + \frac{d}{d\eta}(U - iV)$$

$$= \frac{dT}{d\xi} + \frac{dW}{d\eta}.$$

Hence

$$(\lambda' + \mu)\frac{d}{d\eta}\left(\frac{dT}{d\xi} + \frac{dW}{d\eta}\right) + 2\mu\frac{d^2T}{d\xi\, d\eta} = 0,$$

$$(\lambda' + \mu)\frac{d}{d\xi}\left(\frac{dT}{d\xi} + \frac{dW}{d\eta}\right) + 2\mu\frac{d^2W}{d\xi\, d\eta} = 0.$$

From these, by simple integration,

$$\left.\begin{aligned}(\lambda' + 3\mu)\frac{d\mathrm{T}}{d\xi} + (\lambda' + \mu)\frac{d\mathrm{W}}{d\eta} &= \phi'(\xi)\\(\lambda' + \mu)\frac{d\mathrm{T}}{d\xi} + (\lambda' + 3\mu)\frac{d\mathrm{W}}{d\eta} &= \chi'(\eta)\end{aligned}\right\} \text{ where } \phi'(\xi),\ \chi'(\eta) \text{ are arbitrary functions,}$$

whence

$$\frac{d\mathrm{T}}{d\xi} = \frac{1}{4\mu}\{\phi'(\xi) - \chi'(\eta)\} + \frac{1}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\},$$

$$\frac{d\mathrm{W}}{d\eta} = \frac{1}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\} - \frac{1}{4\mu}\{\phi'(\xi) - \chi'(\eta)\},$$

or

$$\mathrm{T} = \frac{1}{4\mu}[\phi(\xi) - \xi\chi'(\eta)] + \frac{1}{4(\lambda' + 2\mu)}[\phi(\xi) + \xi\chi'(\eta)] + \mathrm{F}(\eta) \quad . \quad . \quad (19),$$

$$\mathrm{W} = \frac{1}{4(\lambda' + 2\mu)}[\eta\phi'(\xi) + \chi(\eta)] - \frac{1}{4\mu}[\eta\phi'(\xi) - \chi(\eta)] + \mathrm{G}(\xi) \quad . \quad . \quad (20),$$

where $\mathrm{F}(\eta)$, $\mathrm{G}(\xi)$ are again arbitrary functions.

Hence U and V can be found almost immediately. Writing

$$\mathrm{F}(\eta) - \frac{1}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}\eta\chi'(\eta) = \mathrm{F}_1(\eta),$$

$$\mathrm{G}(\xi) - \frac{1}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}\xi\phi'(\xi) = \mathrm{G}_1(\xi),$$

we have

$$\mathrm{U} = \frac{\lambda' + 3\mu}{8\mu(\lambda' + 2\mu)}\{\phi(\xi) + \chi(\eta)\} + \frac{i}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}y(\phi'(\xi) - \chi'(\eta)) + \tfrac{1}{2}(\mathrm{F}_1(\eta) + \mathrm{G}_1(\xi)) \quad . \quad (21),$$

$$\mathrm{V} = \frac{i(\lambda' + 3\mu)}{8\mu(\lambda' + 2\mu)}\{\chi(\eta) - \phi(\xi)\} - \frac{1}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}y(\chi'(\eta) + \phi'(\xi)) + \frac{i}{2}\{\mathrm{G}_1(\xi) - \mathrm{F}_1(\eta)\} \quad . \quad (22),$$

from which we obtain easily

$$\mathrm{P} = \frac{3(\lambda' + \mu)}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\} + \frac{\lambda' + \mu}{2(\lambda' + 2\mu)}iy\{\phi''(\xi) - \chi''(\eta)\} + \mu\mathrm{G}'_1(\xi) + \mu\mathrm{F}'_1(\eta),$$

$$\mathrm{Q} = \frac{\lambda' + \mu}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\} - \frac{\lambda' + \mu}{2(\lambda' + 2\mu)}iy(\phi''(\xi) - \chi''(\eta)) - \mu\mathrm{G}'_1(\xi) - \mu\mathrm{F}'_1(\eta),$$

$$\mathrm{S} = -\frac{(\lambda' + \mu)}{2(\lambda' + 2\mu)}y\{\phi''(\xi) + \chi''(\eta)\} + \tfrac{1}{4}\frac{\lambda' + \mu}{\lambda' + 2\mu}i(\phi'(\xi) - \chi'(\eta)) + \mu i\{\mathrm{G}'_1(\xi) - \mathrm{F}_1'(\eta)\},$$

and these last may be put into the simpler form

$$\mathrm{P} = \left(\tfrac{3}{4}\frac{\lambda' + \mu}{\lambda' + 2\mu}\frac{d}{dx} + \frac{\lambda' + \mu}{2(\lambda' + 2\mu)}y\frac{d^2}{dx\,dy}\right)\{\phi(\xi) + \chi(\eta)\} + \mu\frac{d}{dx}\{\mathrm{G}_1(\xi) + \mathrm{F}_1(\eta)\} \quad . \quad (23),$$

$$Q=\left(\tfrac{1}{4}\frac{\lambda'+\mu}{\lambda'+2\mu}\frac{d}{dx}-\frac{\lambda'+\mu}{2(\lambda'+2\mu)}y\frac{d^2}{dxdy}\right)\{\phi(\xi)+\chi(\eta)\}-\mu\frac{d}{dx}\{G_1(\xi)+F_1(\eta)\}\quad . \quad (24),$$

$$S=\left(-\tfrac{1}{2}\frac{\lambda'+\mu}{\lambda'+2\mu}y\frac{d^2}{dx^2}+\tfrac{1}{4}\frac{\lambda'+\mu}{\lambda'+2\mu}\frac{d}{dy}\right)\{\phi(\xi)+\chi(\eta)\}+\mu\frac{d}{dy}\{G_1(\xi)+F_1(\eta)\}\quad . \quad (25),$$

which have the advantage of not containing imaginaries if $\phi(\xi)+\chi(\eta)$, $G_1(\xi)+F_1(\eta)$ are real.

§ 5. *Solution involving Hyperbolic and Circular Functions.*

Assume now for the arbitrary functions the following typical forms

$$\phi(\xi)=A\sin m\xi+iB\cos m\xi+E\cos m\xi+iF\sin m\xi,$$

$$\chi(\eta)=A\sin m\eta-iB\cos m\eta+E\cos m\eta-iF\sin m\eta,$$

$$G_1(\xi)=C\sin m\xi+iD\cos m\xi+G\cos m\xi+iH\sin m\xi,$$

$$F_1(\eta)=C\sin m\eta-iD\cos m\eta+G\cos m\eta-iH\sin m\eta,$$

so that

$$\phi(\xi)+\chi(\eta)=2\sin mx\,(A\cosh my+B\sinh my)$$
$$+\,2\cos mx\,(E\cosh my-F\sinh my),$$

$$\phi(\xi)-\chi(\eta)=2i\cos mx\,(A\sinh my+B\cosh my)$$
$$-\,2i\sin mx\,(E\sinh my-F\cosh my),$$

$$G_1(\xi)+F_1(\eta)=2\sin mx\,(C\cosh my+D\sinh my)$$
$$+\,2\cos mx\,(G\cosh my-H\sinh my),$$

$$G_1(\xi)-F_1(\eta)=2i\cos mx\,(C\sinh my+D\cosh my)$$
$$-\,2i\sin mx\,(G\sinh my-H\cosh my).$$

Whence from (23), (24), (25) we get after some reductions

$$P=\cos mx\left\{\begin{matrix}(3A'+C')\cosh my+(3B'+D')\sin my\\ +2my\,(A'\sinh my+B'\cosh my)\end{matrix}\right\}$$
$$+\sin mx\left\{\begin{matrix}-(3E'+G')\cosh my+(3F'+H')\sinh my\\ +2my\,(-E'\sinh my+F'\cosh my)\end{matrix}\right\}\quad . \quad . \quad (26),$$

$$Q=\cos mx\left\{\begin{matrix}(A'-C')\cosh my+(B'-D')\sinh my\\ -2my\,(A'\sinh my+B'\cosh my)\end{matrix}\right\}$$
$$+\sin mx\left\{\begin{matrix}-(E'-G')\cosh my+(F'-H')\sinh my\\ -2my\,(-E'\sinh my+F'\cosh my)\end{matrix}\right\}\quad . \quad . \quad (27),$$

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$$\mathrm{S} = \sin mx \left\{ \begin{array}{c} (\mathrm{A}' + \mathrm{C}') \sinh my + (\mathrm{B}' + \mathrm{D}') \cosh my \\ + 2my\, (\mathrm{A}' \cosh my + \mathrm{B}' \sinh my) \end{array} \right\}$$

$$+ \cos mx \left\{ \begin{array}{c} (\mathrm{E}' + \mathrm{G}') \sinh my - (\mathrm{H}' + \mathrm{F}') \cosh my \\ + 2my\, (\mathrm{E}' \cosh my - \mathrm{F}' \sinh my) \end{array} \right\} \quad . \quad . \quad . \quad (28),$$

where $\mathrm{A}' = \frac{m}{2} \frac{\lambda' + \mu}{\lambda' + 2\mu} \mathrm{A}$, $\mathrm{B}' = \frac{m}{2} \frac{\lambda' + \mu}{\lambda' + 2\mu} \mathrm{B}$, $\mathrm{E}' = \frac{m}{2} \frac{\lambda' + \mu}{\lambda' + 2\mu} \mathrm{E}$, $\mathrm{F}' = \frac{m}{2} \frac{\lambda' + \mu}{\lambda' + 2\mu} \mathrm{F}$, $\mathrm{C}' = 2\mu m\mathrm{C}$, $\mathrm{D}' = 2\mu m\mathrm{D}$, $\mathrm{G}' = 2\mu m\mathrm{G}$, $\mathrm{H}' = 2\mu m\mathrm{H}$, and the expressions for the mean displacements come out to be

$$\mathrm{U} = \sin mx \left[\begin{array}{c} \frac{1}{2m\mu} \left\{ \frac{\lambda' + 3\mu}{\lambda' + \mu} (\mathrm{A}' \cosh my + \mathrm{B}' \sinh my) + \mathrm{C}' \cosh my + \mathrm{D}' \sinh my \right\} \\ + \frac{1}{\mu}\, y\, (\mathrm{A}' \sinh my + \mathrm{B}' \cosh my) \end{array} \right]$$

$$+ \cos mx \left[\begin{array}{c} \frac{1}{2m\mu} \left\{ \frac{\lambda' + 3\mu}{\lambda' + \mu} (\mathrm{E}' \cosh my - \mathrm{F}' \sinh my) + \mathrm{G}' \cosh my - \mathrm{H}' \sinh my \right\} \\ + \frac{y}{\mu} (\mathrm{E}' \sinh my - \mathrm{F}' \cosh my) \end{array} \right] \quad (29).$$

$$\mathrm{V} = \cos mx \left[\begin{array}{c} \frac{1}{2m\mu} \left\{ \frac{\lambda' + 3\mu}{\lambda' + \mu} (\mathrm{A}' \sinh my + \mathrm{B}' \cosh my) - \mathrm{C}' \sinh my - \mathrm{D}' \cosh my \right\} \\ - \frac{y}{\mu} (\mathrm{A}' \cosh my + \mathrm{B}' \sinh my) \end{array} \right]$$

$$+ \sin mx \left[\begin{array}{c} \frac{1}{2m\mu} \left\{ \frac{\lambda' + 3\mu}{\lambda' + \mu} (-\mathrm{E}' \sinh my + \mathrm{F}' \cosh my) + \mathrm{G}' \sinh my - \mathrm{H}' \cosh my \right\} \\ + \frac{y}{\mu} (\mathrm{E}' \cosh my - \mathrm{F}' \sinh my) \end{array} \right] (30).$$

§ 6. *Determination of the Arbitrary Constants from the Stress Conditions over the Faces $y = \pm b$.*

We shall suppose that the mean stresses Q and S are given arbitrarily over the top and bottom surfaces $y = \pm b$. Expanding these in Fourier series, we have, say :

$$\left. \begin{array}{l} [\mathrm{Q}]_{y=+b} = \alpha_0 + \Sigma\alpha_n \cos mx + \Sigma\gamma_n \sin mx \\ [\mathrm{Q}]_{y=-b} = \beta_0 + \Sigma\beta_n \cos mx + \Sigma\delta_n \sin mx \\ [\mathrm{S}]_{y=+b} = \zeta_0 + \Sigma\zeta_n \cos mx + \Sigma\kappa_n \sin mx \\ [\mathrm{S}]_{y=-b} = \theta_0 + \Sigma\theta_n \cos mx + \Sigma\nu_n \sin mx \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (31),$$

where α_n, β_n, γ_n, δ_n, ζ_n, θ_n, κ_n, ν_n are known constants, and $m = n\pi/a$ where n is any positive integer.

Now, if we take expressions (27) and (28) and equate them, for $y = \pm b$, to the expressions (31), we obtain eight typical equations for the constants which, when combined in pairs, may be written in the simpler form:

$$\left.\begin{aligned}(\mathrm{A}'-\mathrm{C}')\cosh mb-2mb\,\mathrm{A}'\sinh mb&=\frac{\alpha_n+\beta_n}{2}\\(\mathrm{A}'+\mathrm{C}')\sinh mb+2mb\,\mathrm{A}'\cosh mb&=\frac{\kappa_n-\nu_n}{2}\end{aligned}\right\}\quad\ldots\quad(32),$$

$$\left.\begin{aligned}(\mathrm{B}'-\mathrm{D}')\sinh mb-2mb\,\mathrm{B}'\cosh mb&=\frac{\alpha_n-\beta_n}{2}\\(\mathrm{B}'+\mathrm{D}')\cosh mb+2mb\,\mathrm{B}'\sinh mb&=\frac{\kappa_n+\nu_n}{2}\end{aligned}\right\}\quad\ldots\quad(33),$$

$$\left.\begin{aligned}(\mathrm{E}'-\mathrm{G}')\cosh mb-2mb\,\mathrm{E}'\sinh mb&=-\frac{\gamma_n+\delta_n}{2}\\(\mathrm{E}'+\mathrm{G}')\sinh mb+2mb\,\mathrm{E}'\cosh mb&=\frac{\zeta_n-\theta_n}{2}\end{aligned}\right\}\quad\ldots\quad(34),$$

$$\left.\begin{aligned}(\mathrm{F}'-\mathrm{H}')\sinh mb-2mb\,\mathrm{F}'\cosh mb&=\frac{\gamma_n-\delta_n}{2}\\(\mathrm{F}'+\mathrm{H}')\cosh mb+2mb\,\mathrm{F}'\sinh mb&=-\frac{\zeta_n+\theta_n}{2}\end{aligned}\right\}\quad\ldots\quad(35).$$

These equations solve in pairs. We find easily

$$\mathrm{A}'=\frac{\alpha_n+\beta_n}{2}\,\frac{\sinh mb}{\sinh 2mb+2mb}+\frac{\kappa_n-\nu_n}{2}\,\frac{\cosh mb}{\sinh 2mb+2mb}\quad\ldots\quad(36),$$

$$\mathrm{C}'=-\frac{\alpha_n+\beta_n}{2}\,\frac{\sinh mb+2mb\cosh mb}{\sinh 2mb+2mb}+\frac{\kappa_n-\nu_n}{2}\,\frac{\cosh mb-2mb\sinh mb}{\sinh 2mb+2mb}\quad.\quad(37),$$

$$\mathrm{B}'=\frac{\alpha_n-\beta_n}{2}\,\frac{\cosh mb}{\sinh 2mb-2mb}+\frac{\kappa_n+\nu_n}{2}\,\frac{\sinh mb}{\sinh 2mb-2mb}\quad\ldots\quad(38),$$

$$\mathrm{D}'=-\frac{\alpha_n-\beta_n}{2}\,\frac{\cosh mb+2mb\sinh mb}{\sinh 2mb-2mb}+\frac{\kappa_n+\nu_n}{2}\,\frac{\sinh mb-2mb\cosh mb}{\sinh 2mb-2mb}\quad.\quad(39),$$

$$\mathrm{E}'=-\frac{\gamma_n+\delta_n}{2}\,\frac{\sinh mb}{\sinh 2mb+2mb}+\frac{\zeta_n-\theta_n}{2}\,\frac{\cosh mb}{\sinh 2mb+2mb}\quad\ldots\quad(40),$$

$$\mathrm{G}'=\frac{\gamma_n+\delta_n}{2}\,\frac{\sinh mb+2mb\cosh mb}{\sinh 2mb+2mb}+\frac{\zeta_n-\theta_n}{2}\,\frac{\cosh mb-2mb\sinh mb}{\sinh 2mb+2mb}\quad.\ .\quad(41),$$

$$\mathrm{F}'=\frac{\gamma_n-\delta_n}{2}\,\frac{\cosh mb}{\sinh 2mb-2mb}-\frac{\zeta_n+\theta_n}{2}\,\frac{\sinh mb}{\sinh 2mb-2mb}\quad\ldots\quad(42),$$

$$\mathrm{H}'=-\frac{\gamma_n-\delta_n}{2}\,\frac{\cosh mb+2mb\sinh mb}{\sinh 2mb-2mb}-\frac{\zeta_n+\theta_n}{2}\,\frac{\sinh mb-2mb\cosh mb}{\sinh 2mb-2mb}\quad.\quad(43),$$

where in the above n corresponds to a positive integer.

The case where $n = 0$ has to be investigated separately.

§ 7. *Expressions for the Displacements and Stresses.*

Substituting the values of the constants found above into the equations (26)–(30), we obtain for the mean displacements U, V and for the mean stresses, P, Q, S the following values, in so far as we merely consider the terms corresponding to $n =$ a positive integer :—

$$U = \sum_1^\infty \left[\frac{(\alpha_n + \beta_n)\left\{\frac{1}{\lambda' + \mu}\sinh mb - \frac{1}{\mu} mb \cosh mb\right\} + (\kappa_n - \nu_n)\left\{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\cosh mb - \frac{1}{\mu} mb \sinh mb\right\}}{2m(\sinh 2mb + 2mb)} \right] \cosh my \sin mx$$

$$+ \sum_1^\infty \left[\frac{(\alpha_n - \beta_n)\left\{\frac{1}{\lambda' + \mu}\cosh mb - \frac{1}{\mu} mb \sinh mb\right\} + (\kappa_n + \nu_n)\left\{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\sinh mb - \frac{1}{\mu} mb \cosh mb\right\}}{2m(\sinh 2mb - 2mb)} \right] \sinh my \sin mx$$

$$+ \sum_1^\infty \left\{ \frac{\alpha_n + \beta_n}{2\mu} \frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\kappa_n - \nu_n}{2\mu} \frac{\cosh mb}{\sinh 2mb + 2mb} \right\} y \sinh my \sin mx$$

$$+ \sum_1^\infty \left\{ \frac{\alpha_n - \beta_n}{2\mu} \frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\kappa_n + \nu_n}{2\mu} \frac{\sinh mb}{\sinh 2mb - 2mb} \right\} y \cosh my \sin mx$$

$$+ \sum_1^\infty \left[\frac{-(\gamma_n + \delta_n)\left\{\frac{1}{\lambda' + \mu}\sinh mb - \frac{1}{\mu} mb \cosh mb\right\} + (\zeta_n - \theta_n)\left\{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\cosh mb - \frac{1}{\mu} mb \sinh mb\right\}}{2m(\sinh 2mb + 2mb)} \right] \cosh my \cos mx$$

$$+ \sum_1^\infty \left[\frac{-(\gamma_n - \delta_n)\left\{\frac{1}{\lambda' + \mu}\cosh mb - \frac{1}{\mu} mb \sinh mb\right\} + (\zeta_n + \theta_n)\left\{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\sinh mb - \frac{1}{\mu} mb \cosh mb\right\}}{2m(\sinh 2mb - 2mb)} \right] \sinh my \cos mx$$

$$+ \sum_1^\infty \left\{ \frac{-(\gamma_n + \delta_n)}{2\mu} \frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{(\zeta_n - \theta_n)}{2\mu} \frac{\cosh mb}{\sinh 2mb + 2mb} \right\} y \sinh my \cos mx$$

$$+ \sum_1^\infty \left\{ \frac{-(\gamma_n - \delta_n)}{2\mu} \frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{(\zeta_n + \theta_n)}{2\mu} \frac{\sinh mb}{\sinh 2mb - 2mb} \right\} y \cosh my \cos mx \qquad (44).$$

$$
\begin{aligned}
V = {} & \sum_1^\infty \frac{\left[\begin{array}{l}(\alpha_n + \beta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb + \dfrac{mb}{\mu}\cosh mb\right\} \\ \quad + (\kappa_n - \nu_n)\left\{\dfrac{1}{\lambda' + \mu}\cosh mb + \dfrac{mb}{\mu}\sinh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb + 2mb)} \sinh my \cos mx \\
& + \sum_1^\infty \frac{\left[\begin{array}{l}(\alpha_n - \beta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb + \dfrac{mb}{\mu}\sinh mb\right\} \\ \quad + (\kappa_n + \nu_n)\left\{\dfrac{1}{\lambda' + \mu}\sinh mb + \dfrac{mb}{\mu}\cosh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb - 2mb)} \cosh my \cos mx \\
& - \sum_1^\infty \left\{\frac{\alpha_n + \beta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\kappa_n - \nu_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb + 2mb}\right\} y \cosh my \cos mx \\
& - \sum_1^\infty \left\{\frac{\alpha_n - \beta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\kappa_n + \nu_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb - 2mb}\right\} y \sinh my \cos mx \\
& + \sum_1^\infty \frac{\left[\begin{array}{l}(\gamma_n + \delta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb + \dfrac{mb}{\mu}\cosh mb\right\} \\ \quad - (\zeta_n - \theta_n)\left\{\dfrac{\cosh mb}{\lambda' + \mu} + \dfrac{mb}{\mu}\sinh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb + 2mb)} \sinh my \sin mx \\
& + \sum_1^\infty \frac{\left[\begin{array}{l}(\gamma_n - \delta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb + \dfrac{mb}{\mu}\sinh mb\right\} \\ \quad - (\zeta_n + \theta_n)\left\{\dfrac{\sinh mb}{\lambda' + \mu} + \dfrac{mb}{\mu}\cosh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb - 2mb)} \cosh my \sinh mx \\
& + \sum_1^\infty \left\{-\frac{\gamma_n + \delta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\zeta_n - \theta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb + 2mb}\right\} y \cosh my \sin mx \\
& + \sum_1^\infty \left\{-\frac{\gamma_n - \delta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\zeta_n + \theta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb - 2mb}\right\} y \sinh my \sin mx
\end{aligned} \qquad (45).
$$

$$
\begin{aligned}
P = {} & \sum_1^\infty \frac{(\alpha_n + \beta_n)(\sinh mb - mb\cosh mb) + (\kappa_n - \nu_n)(2\cosh mb - mb\sinh mb)}{\sinh 2mb + 2mb} \cosh my \cos mx \\
& + \sum_1^\infty \frac{(\alpha_n - \beta_n)(\cosh mb - mb\sinh mb) + (\kappa_n + \nu_n)(2\sinh mb - mb\cosh mb)}{\sinh 2mb - 2mb} \sinh my \cos mx \\
& + \sum_1^\infty \frac{(\alpha_n + \beta_n)\sinh mb + (\kappa_n - \nu_n)\cosh mb}{\sinh 2mb + 2mb}\, my \sinh my \cos mx \\
& + \sum_1^\infty \frac{(\alpha_n - \beta_n)\cosh mb + (\kappa_n + \nu_n)\sinh mb}{\sinh 2mb - 2mb}\, my \cosh my \cos mx \\
& + \sum_1^\infty \frac{(\gamma_n + \delta_n)(\sinh mb - mb\cosh mb) - (\zeta_n - \theta_n)(2\cosh mb - mb\sinh mb)}{\sinh 2mb + 2mb} \cosh my \sin mx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_1^\infty \frac{(\gamma_n-\delta_n)(\cosh mb - mb\sinh mb) - (\zeta_n+\theta_n)(2\sinh mb - mb\cosh mb)}{\sinh 2mb - 2mb}\sinh my \sin mx \\
&+\sum_1^\infty \frac{(\gamma_n+\delta_n)\sinh mb - (\zeta_n-\theta_n)\cosh mb}{\sinh 2mb + 2mb}\, my\sinh my \sin mx \\
&+\sum_1^\infty \frac{(\gamma_n-\delta_n)\cosh mb - (\zeta_n+\theta_n)\sinh mb}{\sinh 2mb - 2mb}\, my\cosh my \sin mx \quad . \quad . \quad . \quad . \quad (46).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Q} = &\sum_1^\infty \frac{(\alpha_n+\beta_n)(\sinh mb + mb\cosh mb) + (\kappa_n-\nu_n)\, mb\sinh mb}{\sinh 2mb + 2mb}\cosh my \cos mx \\
&+\sum_1^\infty \frac{(\alpha_n-\beta_n)(\cosh mb + mb\sinh mb) + (\kappa_n+\nu_n)\, mb\cosh mb}{\sinh 2mb - 2mb}\sinh my \cos mx \\
&-\sum_1^\infty \frac{(\alpha_n+\beta_n)\sinh mb + (\kappa_n-\nu_n)\cosh mb}{\sinh 2mb + 2mb}\, my\sinh my \cos mx \\
&-\sum_1^\infty \frac{(\alpha_n-\beta_n)\cosh mb + (\kappa_n+\nu_n)\sinh mb}{\sinh 2mb - 2mb}\, my\cosh my \cos mx \\
&+\sum_1^\infty \frac{(\gamma_n+\delta_n)(\sinh mb + mb\cosh mb) - (\zeta_n-\theta_n)\, mb\sinh mb}{\sinh 2mb + 2mb}\cosh my \sin mx \\
&+\sum_1^\infty \frac{(\gamma_n-\delta_n)(\cosh mb + mb\sinh mb) - (\zeta_n+\theta_n)\, mb\cosh mb}{\sinh 2mb - 2mb}\sinh my \sin mx \\
&-\sum_1^\infty \frac{(\gamma_n+\delta_n)\sinh mb - (\zeta_n-\theta_n)\cosh mb}{\sinh 2mb + 2mb}\, my\sinh my \sin mx \\
&-\sum_1^\infty \frac{(\gamma_n-\delta_n)\cosh mb - (\zeta_n+\theta_n)\sinh mb}{\sinh 2mb - 2mb}\, my\cosh my \sin mx \quad . \quad . \quad . \quad . \quad (47).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{S} = &\sum_1^\infty \frac{-(\alpha_n+\beta_n)\, mb\cosh mb + (\kappa_n-\nu_n)(\cosh mb - mb\sinh mb)}{\sinh 2mb + 2mb}\sinh my \sin mx \\
&+\sum_1^\infty \frac{-(\alpha_n-\beta_n)\, mb\sinh mb + (\kappa_n+\nu_n)(\sinh mb - mb\cosh mb)}{\sinh 2mb - 2mb}\cosh my \sin mx \\
&+\sum_1^\infty \frac{(\alpha_n+\beta_n)\sinh mb + (\kappa_n-\nu_n)\cosh mb}{\sinh 2mb + 2mb}\, my\cosh my \sin mx \\
&+\sum_1^\infty \frac{(\alpha_n-\beta_n)\cosh mb + (\kappa_n+\nu_n)\sinh mb}{\sinh 2mb - 2mb}\, my\sinh my \sin mx \\
&+\sum_1^\infty \frac{(\gamma_n+\delta_n)\, mb\cosh mb + (\zeta_n-\theta_n)(\cosh mb - mb\sinh mb)}{\sinh 2mb + 2mb}\sinh my \cos mx \\
&+\sum_1^\infty \frac{(\gamma_n-\delta_n)\, mb\sinh mb + (\zeta_n+\theta_n)(\sinh mb - mb\cosh mb)}{\sinh 2mb - 2mb}\cosh my \cos mx \\
&+\sum_1^\infty \frac{-(\gamma_n+\delta_n)\sinh mb + (\zeta_n-\theta_n)\cosh mb}{\sinh 2mb + 2mb}\, my\cosh my \cos mx \\
&+\sum_1^\infty \frac{-(\gamma_n-\delta_n)\cosh mb + (\zeta_n+\theta_n)\sinh mb}{\sinh 2mb - 2mb}\, my\sinh my \cos mx \quad . \quad . \quad . \quad . \quad . \quad (48).
\end{aligned}
$$

§ 8. *Conditions at the Two Ends* $x = \pm a$.

It is, however, impossible to satisfy fully the conditions over the two ends $x = \pm a$. These would require that P and S should have given values over these ends. If, however, a is so large that, at a long distance from the ends, the effect of any self-equilibrating system of stress over these same ends may be neglected, then we need only consider *total* terminal conditions at $x = \pm a$.

These conditions will involve

(i.) The total tension $\mathrm{T} = \int_{-b}^{b} \mathrm{P}\, dy$ across either end.

(ii.) The total shear $\bar{\mathrm{S}} = \int_{-b}^{b} \mathrm{S}\, dy$ across either end.

(iii.) The bending moment $\mathrm{M} = -\int_{-b}^{b} \mathrm{P}y\, dy$ across either end.

I now propose to calculate the quantities T, $\bar{\mathrm{S}}$ and M for that part of the solution which has been given in the last section.

I find, after reduction,

$$(\mathrm{T})_{+a} = (\mathrm{T})_{-a} = \sum_{1}^{\infty} \frac{\cos ma}{m} (\kappa_n - \nu_n) \quad \text{. . . . . . (49).}$$

$$(\bar{\mathrm{S}})_{a} = (\bar{\mathrm{S}})_{-a} = \sum_{1}^{\infty} (\gamma_n - \delta_n) \frac{\cos ma}{m} \quad \text{. . . . . (50).}$$

$$-(\mathrm{M})_{+a} = -(\mathrm{M})_{-a} = \sum_{1}^{\infty} \frac{(\alpha_n - \beta_n)}{m^2} \cos ma + \sum_{1}^{\infty} \frac{b}{m} (\kappa_n + \nu_n) \cos ma \quad \text{. (51).}$$

Now we can always adjust M and T so as to be zero, for the solutions for a uniform tension and a uniform bending moment, viz.:—

$$\left. \begin{aligned} \mathrm{U} &= \frac{\mathrm{T}x}{2b\mathrm{E}} - \frac{3\mathrm{M}xy}{2b^3\mathrm{E}} \\ \mathrm{V} &= \frac{\eta \mathrm{T} y}{2b\mathrm{E}} + \frac{3\mathrm{M}}{2b^3\mathrm{E}} \frac{x^2 - \eta y^2}{2} \end{aligned} \right\} \quad \text{. . . . . (52)}$$

(where $\eta = -\frac{1}{2}\lambda/(\lambda + \mu)$ and E is YOUNG's Modulus), produce no stress across the faces $y = \pm b$, and therefore such solutions can always be arbitrarily superimposed. They correspond to stresses which are *transmitted* from the ends; and we shall find that it is necessary, in various cases, to add such solutions in order to satisfy the end conditions, which are not necessarily satisfied by the series merely involving circular functions.

§ 9. *Part of the Solutions Corresponding to the Terms* α_0, β_0, ζ_0, θ_0.

In the first place it is obvious, having regard to the conditions of rigid equilibrium, that if the ends $x = \pm a$ are free from stress, then α_0 must $= \beta_0$. If $\alpha_0 \neq \beta_0$ we must have a shear over the two ends in order to balance the excess of the pressure on the one side over the pressure on the other side, and this will require special investigation. The solution arising from such conditions is discussed in §§ 39–40. For the present let us confine ourselves to $\alpha_0 = \beta_0$. This corresponds to a uniform traction along the axis y and introduces the following additional terms :—

$$\left.\begin{aligned} &U = \frac{\eta\alpha_0 x}{E}, && V = \frac{\alpha_0 y}{E} \\ &P = 0, \quad Q = \alpha_0, && S = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

Now turning to the terms in ζ_0 and θ_0, it is easy to verify that the additional terms

$$\left.\begin{aligned} &U = \frac{-(\lambda' + 2\mu)}{16\mu(\lambda' + \mu)}(\zeta_0 - \theta_0)\frac{x^2}{b} + \frac{3\lambda' + 4\mu}{16\mu(\lambda' + \mu)}(\zeta_0 - \theta_0)\frac{y^2}{b} \\ &V = \frac{\lambda'}{8\mu(\lambda' + \mu)}(\zeta_0 - \theta_0)\frac{xy}{b} \end{aligned}\right.$$

and therefore

$$\left.\begin{aligned} Q = 0, \qquad P = -\frac{\zeta_0 - \theta_0}{2b}x, \qquad S = \frac{\zeta_0 - \theta_0}{2b}y \end{aligned}\right\} \quad . \quad . \quad (54),$$

satisfy the conditions that S shall have constant values over the two boundaries $y = \pm b$, these values being equal in magnitude and opposite in sign. The effect of these shears is balanced by the pressure and tension $(\zeta_0 - \theta_0)\,a/2b$ over the two ends, and the conditions of rigid equilibrium are satisfied.

Finally, if we have equal shears over the boundaries, the sign being the same (so that the external impressed forces act in *opposite* directions), the solution

$$\left.\begin{aligned} &U = \frac{1}{4\mu}(\zeta_0 + \theta_0)\,y, && V = \frac{1}{4\mu}(\zeta_0 + \theta_0)\,x \\ &P = 0, \qquad Q = 0, && S = \tfrac{1}{2}(\zeta_0 + \theta_0) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55)$$

will satisfy all conditions over the boundaries $y = \pm b$, and will introduce over the boundaries $x = \pm a$ a system of shear necessary to maintain rigid equilibrium.

Adding together the solutions (54) and (55), we find that the conditions $Q = 0$ over $y = \pm b$, $S = \zeta_0$ over $y = +b$, $S = \theta_0$ over $y = -b$ are all satisfied.

This completes the solution of the problem proposed, with the exception of the case $\alpha_0 \neq \beta_0$, which can be reduced to the problem of a beam uniformly loaded along the top and free along the bottom, the load being taken by shears over the ends.

PART II.

Discussion of the General Solution when the Forces on the Beam are Purely Normal and are Symmetrical about $x = 0$.

§ 10. *Expressions for the Stresses and Displacements.*

If the forces are purely normal, and if the solution is to be even in x, then the γ, δ, ζ, θ, κ, ν terms disappear.

Further, we have the additional condition that, over the ends $x = \pm a$, $\mathrm{T} = 0$, $\bar{\mathrm{S}} = 0$, $\mathrm{M} = 0$; by introducing suitable terms of the form (52) we can satisfy this last condition, and we finally obtain

$$\left.\begin{aligned}
\mathrm{U} = {} & \frac{\eta\alpha_0 x}{\mathrm{E}} - \frac{3xy}{2b^3\mathrm{E}} \sum_1^\infty (\alpha_n - \beta_n) \frac{\cos ma}{m^2} \\
& + \sum_1^\infty \frac{(\alpha_n + \beta_n)}{2m} \frac{\left(\dfrac{1}{\lambda' + \mu} \sinh mb - \dfrac{1}{\mu} mb \cosh mb\right)}{\sinh 2mb + 2mb} \cosh my \sin mx \\
& + \sum_1^\infty \frac{(\alpha_n - \beta_n)}{2m} \frac{\left(\dfrac{1}{\lambda' + \mu} \cosh mb - \dfrac{1}{\mu} mb \sinh mb\right)}{\sinh 2mb - 2mb} \sinh my \sin mx \\
& + \sum_1^\infty \frac{(\alpha_n + \beta_n)}{2\mu} \frac{y \sinh mb \sinh my \sin mx}{\sinh 2mb + 2mb} + \sum_1^\infty \frac{(\alpha_n - \beta_n)}{2\mu} \frac{y \cosh mb \cosh my \sin mx}{\sinh 2mb - 2mb} \\
\mathrm{V} = {} & \frac{\alpha_0 y}{\mathrm{E}} + \frac{3}{2b^3\mathrm{E}} \left(\frac{x^2 - \eta y^2}{2}\right) \sum_1^\infty (\alpha_n - \beta_n) \frac{\cos ma}{m^2} + \mathrm{B} \\
& + \sum_1^\infty \left(\frac{\alpha_n + \beta_n}{2m}\right) \frac{\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right) \sinh mb + \dfrac{1}{\mu} mb \cosh mb\right\}}{\sinh 2mb + 2mb} \sinh my \cos mx \\
& + \sum_1^\infty \left(\frac{\alpha_n - \beta_n}{2m}\right) \frac{\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right) \cosh mb + \dfrac{1}{\mu} mb \sinh mb\right\}}{\sinh 2mb - 2mb} \cosh my \cos mx \\
& - \sum_1^\infty \left(\frac{\alpha_n + \beta_n}{2\mu}\right) \frac{y \sinh mb \cosh my \cos mx}{\sinh 2mb + 2mb} - \sum_1^\infty \left(\frac{\alpha_n - \beta_n}{2\mu}\right) \frac{y \cosh mb \sinh my \cos mx}{\sinh 2mb - 2mb}
\end{aligned}\right\} \quad (56),$$

where B is an arbitrary constant to be determined from some condition of fixing. It merely corresponds to a total vertical displacement of the beam.

$$\left.\begin{aligned}
\mathrm{P} = {} & -\frac{3y}{2b^3} \sum_1^\infty (\alpha_n - \beta_n) \frac{\cos ma}{m^2} + \sum_1^\infty (\alpha_n + \beta_n) \frac{\sinh mb - mb \cosh mb}{\sinh 2mb + 2mb} \cosh my \cos mx \\
& + \sum_1^\infty (\alpha_n - \beta_n) \frac{\cosh mb - mb \sinh mb}{\sinh 2mb - 2mb} \sinh my \cos mx \\
& + \sum_1^\infty (\alpha_n + \beta_n) \frac{my \sinh mb \sinh my \cos mx}{\sinh 2mb + 2mb} + \sum_1^\infty (\alpha_n - \beta_n) \frac{my \cosh mb \cosh my \cos mx}{\sinh 2mb - 2mb}
\end{aligned}\right\} (57),$$

$$\left.\begin{aligned}
Q = \alpha_0 &+ \sum_1^\infty (\alpha_n + \beta_n) \frac{\sinh mb + mb \cosh mb}{\sinh 2mb + 2mb} \cosh my \cos mx \\
&- \sum_1^\infty (\alpha_n + \beta_n) \frac{my \sinh mb \sinh my \cos mx}{\sinh 2mb + 2mb} \\
&+ \sum_1^\infty (\alpha_n - \beta_n) \frac{\cosh mb + mb \sinh mb}{\sinh 2mb - 2mb} \sinh my \cos mx \\
&- \sum_1^\infty (\alpha_n - \beta_n) \frac{my \cosh mb \cosh my \cos mx}{\sinh 2mb - 2mb} \\
S = &- \sum_1^\infty (\alpha_n + \beta_n) \frac{mb \cosh mb \sinh my \sin mx}{\sinh 2mb + 2mb} - \sum_1^\infty (\alpha_n - \beta_n) \frac{mb \sinh mb \cosh my \sin mx}{\sinh 2mb - 2mb} \\
&+ \sum_1^\infty (\alpha_n + \beta_n) \frac{my \sinh mb \cosh my \sin mx}{\sinh 2mb + 2mb} + \sum_1^\infty (\alpha_n - \beta_n) \frac{my \cosh mb \sinh my \sin mx}{\sinh 2mb - 2mb}
\end{aligned}\right\} \quad (57).$$

§ 11. *Approximate Values to which the Expressions of* § 10 *lead when "b" is made very small.*

If b is very small compared with a, so that, even for certain fairly high values of m, mb is still small, we may expand the coefficients in (56) and (57) in powers of mb, and also we may expand cosh my and sinh my in powers of my. This is the method which has been employed by POCHHAMMER ('Crelle's Journal,' vol. 81). I have shown in a previous paper ("On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load," 'Phil. Trans.,' A, vol. 198, pp. 147–233), that such an approximation was valid provided that the original series and each of the approximate series obtained from the various terms in the expansion of the coefficients of cos mx, sin mx (which expansion is supposed carried out only to a limited number of terms) are absolutely and uniformly convergent for the region considered.

Assuming that the values of α_n, β_n are such as to ensure that these conditions are satisfied, let us see what happens when, in the expressions for the displacements U and V, we neglect all terms of order greater than -1 in m.

We find

$$U = -\frac{3xy}{2b^3 E} \sum_1^\infty (\alpha_n - \beta_n) \frac{\cos ma}{m^2} + \sum_1^\infty \left(\frac{1}{\lambda' + \mu} - \frac{1}{\mu}\right) \frac{\alpha_n + \beta_n}{8m} \sin mx$$

$$+ \sum_1^\infty \frac{\alpha_n - \beta_n}{2} \frac{\left\{\frac{1}{\lambda' + \mu}\left(1 + \frac{m^2 b^2}{2}\right) - \frac{1}{\mu} m^2 b^2\right\}}{\frac{4}{3} m^3 b^3 \left(1 + \frac{m^2 b^2}{5}\right)} y \left(1 + \frac{m^2 y^2}{6}\right) \sin mx$$

$$+ \sum_1^\infty \frac{\alpha_n - \beta_n}{2\mu} \frac{\left\{1 + \frac{m^2 b^2}{2} + \frac{m^2 y^2}{2}\right\}}{\frac{4}{3} m^3 b^3 \left(1 + \frac{m^2 b^2}{5}\right)} y \sin mx$$

$$= -\frac{3xy}{2b^3\mathrm{E}}\sum_1^\infty(\alpha_n-\beta_n)\frac{\cos ma}{m^2} - \frac{\lambda'}{8\mu(\lambda'+\mu)}\sum_1^\infty\frac{\alpha_n+\beta_n}{m}\sin mx$$

$$+\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3y}{8b^3}\sum_1^\infty\frac{\alpha_n-\beta_n}{m^3}\sin mx$$

$$+\frac{3y}{8b}\left\{\frac{3\lambda'+4\mu}{6\mu(\lambda'+\mu)}\frac{y^2}{b^2}-\frac{7\lambda'+4\mu}{10\mu(\lambda'+\mu)}\right\}\sum_1^\infty\frac{\alpha_n-\beta_n}{m}\sin mx \quad . \quad . \quad . \quad . \quad . \quad (58).$$

$$\mathrm{V} = \frac{3}{2b^3\mathrm{E}}\frac{x^2-\eta y^2}{2}\sum_1^\infty(\alpha_n-\beta_n)\frac{\cos ma}{m^2} - \sum_1^\infty\frac{(\alpha_n-\beta_n)}{2\mu}\frac{y^2\cos mx}{\frac{4}{3}m^2b^3}$$

$$+\sum_1^\infty\frac{(\alpha_n-\beta_n)}{2m}\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left(1+\frac{m^2b^2}{2}\right)+\frac{1}{\mu}m^3b^2}{\frac{4}{3}m^2b^2\left(1+\frac{m^2b^2}{5}\right)}\left(1+\frac{m^2y^2}{2}\right)\cos mx$$

$$= \frac{3}{2b^3\mathrm{E}}\frac{x^2-\eta y^2}{2}\sum_1^\infty(\alpha_n-\beta_n)\frac{\cos ma}{m^2}+\frac{3}{8}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{1}{b^3}\sum_1^\infty\frac{\alpha_n-\beta_n}{m^4}\cos mx$$

$$+\frac{3}{8b}\left\{\frac{13\lambda'+16\mu}{10\mu(\lambda'+\mu)}-\frac{\lambda'}{2\mu(\lambda'+\mu)}\frac{y^2}{b^2}\right\}\sum_1^\infty\frac{(\alpha_n-\beta_n)}{m^2}\cos mx \quad . \quad . \quad . \quad . \quad . \quad (59).$$

Now $\sum_1^\infty(\alpha_n-\beta_n)\cos mx = \mathrm{L}$, where L is the difference of stress on the top and bottom, in other words, the transverse load per unit length of the beam.

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m}\sin mx = \int_0^x \mathrm{L}dx = \int_a^x \mathrm{L}dx = -\bar{\mathrm{S}},$$

where $\bar{\mathrm{S}}$ is the total shear at any section.

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos mx - \sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos ma = +\int_a^x\bar{\mathrm{S}}dx = -\mathrm{M},$$

where M is the bending moment at any section.

Integrating again:

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m^3}\sin mx - x\sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos ma = -\int_0^x\mathrm{M}dx$$

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m^4}-\sum_1^\infty\frac{\alpha_n-\beta_n}{m^4}\cos mx-\frac{x^2}{2}\sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos ma = -\int_0^x\left(\int_0^x\mathrm{M}dx\right)dx.$$

Also if $\bar{\mathrm{Q}}$ is the transverse tensile stress at any section $\bar{\mathrm{Q}} = \sum_1^\infty\frac{\alpha_n+\beta_n}{2}\cos mx$, and $\sum_1^\infty\frac{\alpha_n+\beta_n}{2m}\sin mx = \int_0^x\bar{\mathrm{Q}}dx.$

Substituting from the above values into the expressions (58) and (59) for U and V, we find, remembering that $\frac{1}{\lambda'+\mu}+\frac{1}{\mu} = \frac{4}{\mathrm{E}}$ and $\eta = -\lambda'/(\lambda'+2\mu)$,

$$\left.\begin{aligned} \mathrm{U} &= -\frac{3y}{2b^3\mathrm{E}}\int_0^x \mathrm{M}dx - \frac{3y}{8b}\,\bar{\mathrm{S}}\left\{\frac{3\lambda'+4\mu}{6\mu(\lambda'+\mu)}\frac{y^2}{b^2} - \frac{7\lambda'+4\mu}{10\mu(\lambda'+\mu)}\right\} + \frac{\eta}{\mathrm{E}}\int_0^x \bar{\mathrm{Q}}dx \\ \mathrm{V} &= \frac{3}{2b^3\mathrm{E}}\int_0^x\!\!\int_0^x \mathrm{M}dx^2 - \frac{3\mathrm{M}}{8b}\left\{\frac{13\lambda'+16\mu}{10\mu(\lambda'+\mu)} - \frac{\lambda'}{2\mu(\lambda'+\mu)}\frac{y^2}{b^2}\right\} \end{aligned}\right\} \quad (60),$$

dropping a constant in V.

The stresses P, Q, S might be directly deduced from the equations (60) by differentiation. But here we require to be extremely careful, for, y and x being of different orders of magnitude, differentiation with regard to y will not give a term of the same order as differentiation with regard to x. The criterion to be used in this case is this: The series $\mathrm{L} = -d\bar{\mathrm{S}}/dx$ is of order 0 in m, and is therefore among the terms which we have agreed to neglect. Similarly for the series $\bar{\mathrm{Q}}$. In consequence, every time L and $\bar{\mathrm{Q}}$ appear owing to differentiation, they should be neglected if we keep the same order of approximation for the stresses as for the displacements. It will then be found that some terms disappear whose effect is felt in the displacements, as it were, by accumulation.

Keeping this rule in mind, we obtain easily

$$\left.\begin{aligned} \mathrm{P} &= -\frac{3\mathrm{M}y}{2b^3} \\ \mathrm{Q} &= 0 \\ \mathrm{S} &= \frac{3}{4b}\,\bar{\mathrm{S}}\,(b^2 - y^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

Now these are the stresses we should have obtained had we treated that part of the bar as *free*, but subject to a bending moment M and a total shear $\bar{\mathrm{S}}$, transmitted from a distant terminal. Hence we see that, to a first approximation the stress at each point of a bar, whatever the manner of its transverse loading, depends only upon the total bending moment at the section and upon the total shear at the section, and will be given in terms of these by the same formulæ which are valid for a *free* bar subjected to a given couple and shear at its extremities. Similar conclusions follow from the formulæ found by Professor POCHHAMMER in the paper quoted previously.

§ 12. *Analysis of the Approximate Expressions for the Displacements. Shearing Deflection.*

Now if we look at the values (60) we see easily that they are composed of three parts.

(i.) The parts $-\frac{3y}{2b^3\mathrm{E}}\int_0^x \mathrm{M}dx$ of U and $\frac{3}{2b^3\mathrm{E}}\int_0^x\!\!\int_0^x \mathrm{M}dx^2$ of V.

These are what we may call the "Euler-Bernoulli" terms. They correspond to a strain in which cross-sections originally plane remain plane, and the curvature of the elastic line is at all points proportional to the bending moment.

(ii.) The part $\int_0^x \frac{\eta \overline{\mathrm{Q}}}{\mathrm{E}}\, dx$ of U. This corresponds to the lateral contraction of the material under tensions $\overline{\mathrm{Q}}$, and is the same as if each strip of thickness dx and height $2b$ were independently stretched.

(iii.) The terms $-\frac{3y}{8b}\,\overline{\mathrm{S}}\left\{\frac{3\lambda'+4\mu}{6\mu(\lambda'+\mu)}\frac{y^2}{b^2}-\frac{7\lambda'+4\mu}{10\mu(\lambda'+\mu)}\right\}$ of U and

$$-\frac{3\mathrm{M}}{8b}\left\{\frac{13\lambda'+16\mu}{10\mu(\lambda'+\mu)}-\frac{\lambda'}{2\mu(\lambda'+\mu)}\frac{y^2}{b^2}\right\} \text{ of V.}$$

These correspond to a distortion of the cross-sections and to a parabolic distribution of shear.

In the particular case, where the load reduces to a central isolated weight W and the two symmetrical support reactions, the additional terms (iii.) in V are of the form (omitting the constant)

$$\tfrac{3}{8}\frac{\mathrm{W}x}{\mu b}\left\{\frac{13\lambda'+16\mu}{20(\lambda'+\mu)}\right\}+\tfrac{3}{8}\frac{\eta \mathrm{W}(l-x)y^2}{\mathrm{E}b^3} \text{ for } x>0$$

and

$$-\tfrac{3}{8}\frac{\mathrm{W}x}{\mu b}\left\{\frac{13\lambda'+16\mu}{20(\lambda'+\mu)}\right\}+\tfrac{3}{8}\frac{\eta \mathrm{W}(l+x)y^2}{\mathrm{E}b^3} \text{ for } x<0,$$

$2l$ being the distance between the supports.

It might have been supposed that this particular problem would have been capable of solution by breaking up the beam in the middle and treating it as two inverted cantilevers, to each of which we could apply DE SAINT-VENANT'S solution. This, I believe, is often done by engineers.

Now such an attempt is, in strictness, bound to fail, because DE SAINT-VENANT'S solution implies distortion of the cross-section at the fixed end, whereas in the present problem the central cross-section of the beam must necessarily remain plane, from symmetry.

Moreover, we are left in doubt as to the condition of fixing to be adopted. Are we to suppose, with DE SAINT-VENANT, the central element of the terminal cross-section to remain vertical, or, with Professor LOVE ('Theory of Elasticity,' vol. 1, pp. 179–180), the elastic line to be horizontal at the built-in end? In the case of a cantilever the difference is quite immaterial, as it merely amounts to a rigid body displacement. But here we must remember the cantilevers are only fictitiously severed, and the above difference corresponds to an actual sharp bend of the beam in the middle.

It is interesting to compare the true solution with those obtained in this way.

If we assume DE SAINT-VENANT'S fixing condition, we find, for the additional terms in V corresponding to (iii.),

$$\tfrac{3}{8}\frac{\mathrm{W}x}{\mu b}+\tfrac{3}{8}\eta\frac{\mathrm{W}(l-x)y^2}{\mathrm{E}b^3}\text{ for } x>0, \text{ and}$$

$$-\tfrac{3}{8}\frac{\mathrm{W}x}{\mu b}+\tfrac{3}{8}\eta\frac{\mathrm{W}(l+x)y^2}{\mathrm{E}b^3}\text{ for } x<0.$$

The y^2 terms are therefore identical in this and in the true solution, but the first term which represents the additional deflection of the central axis of the beam, and which is sometimes spoken of as the shearing deflection, is less than in the true solution, being $(13\lambda'+16\mu)/20(\lambda'+\mu)$, that is $(42\lambda+32\mu)/(60\lambda+40\mu)$ of that given by the double cantilever solution. This fraction comes to be ·74 for uni-constant isotropy.

If we assume what I have called LOVE'S fixing condition, the shearing deflection disappears entirely.

The true solution shows us, therefore, that it is permissible in this case to use the double cantilever as an artifice to obtain the solution, *provided* we adopt, at the section of fictitious severance, a fixing condition intermediate between those of LOVE and DE SAINT-VENANT, but nearer to the latter. In other words, a central isolated load does actually introduce a sharp bend.

§ 13. *Value of the Deflection when b is not small and the Beam is Doubly Supported.*

Suppose the beam rests on two knife-edge supports A, B (fig. i.) at a distance $2l$ apart, and a weight W is borne by another knife-edge which presses on the upper part of the beam at C.

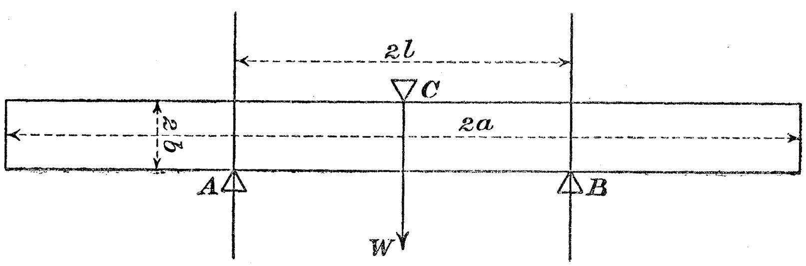

Fig. i.

Then we have $\alpha_0=-\frac{\mathrm{W}}{2a}$, $\alpha_n=-\frac{\mathrm{W}}{a}\,(n\neq 0)$, $\beta_0=\alpha_0$, $\beta_n=-\frac{\mathrm{W}}{a}\cos\frac{n\pi l}{a}$.

The central deflection of the elastic line (what DE SAINT-VENANT calls "la flèche de flexion") is then given by $f=\mathrm{V}_{x=l\,y=0}-\mathrm{V}_{x=0\,y=0}$; substituting for α's and β's in (56), we find

$$f = \frac{3}{4b^3\mathrm{E}}\, l^2 \sum_1^\infty \frac{\cos n\pi}{m^2} \frac{\mathrm{W}}{a}\left(\cos\frac{n\pi l}{a} - 1\right)$$

$$+ \sum_1^\infty \frac{\mathrm{W}}{a}\left(\cos\frac{n\pi l}{a} - 1\right)\frac{1}{2m}\,\frac{\left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh mb + \frac{mb}{\mu}\sinh mb\right\}}{\sinh 2mb - 2mb}\left(\cos\frac{n\pi l}{a} - 1\right). \quad (62).$$

Now the first term can be evaluated. It is $\frac{3}{16}\frac{l^4}{ab^3}\frac{\mathrm{W}}{\mathrm{E}}$. We have therefore

$$f = \frac{3\mathrm{W}l^4}{16\mathrm{E}b^3 a} + \sum_1^\infty \frac{\mathrm{W}}{2n\pi}\,\frac{\left(\frac{1}{\lambda'+\mu}\cosh\frac{n\pi b}{a} + \frac{1}{\mu}\left(\cosh\frac{n\pi b}{a} + \frac{n\pi b}{a}\sinh\frac{n\pi b}{a}\right)\right)}{\left(\sinh\frac{2n\pi b}{a} - \frac{2n\pi b}{a}\right)}\left(1 - \cos\frac{n\pi l}{a}\right)^2.$$

Now let us remove the ends to infinity, that is, make a very large. This will transform the Σ above into a definite integral. It is easily seen that the term under the Σ remains finite and continuous when n is made zero; we may therefore take our limits from 0 to ∞. We then obtain, putting $n\pi b/a = u$, $\pi b/a = du$:

$$f = \int_0^\infty \frac{\mathrm{W}}{2\pi}\,\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u + \frac{u}{\mu}\sinh u}{\sinh 2u - 2u}\left(1 - \cos\frac{ul}{b}\right)^2\frac{du}{u}$$

$$= \frac{2\mathrm{W}}{\pi}\int_0^\infty \frac{\frac{4}{\mathrm{E}}\cosh u + \frac{u}{\mu}\sinh u}{\sinh 2u - 2u}\left(\sin\frac{ul}{2b}\right)^4\frac{du}{u},$$

or writing $l/2b = \lambda_0$,

$$f = \frac{2\mathrm{W}}{\pi}\left(\frac{l}{2b}\right)^4\int_0^\infty \frac{\left(\frac{4u^3\cosh u}{\mathrm{E}} + \frac{u^4\sinh u}{\mu}\right)}{\sinh 2u - 2u}\left(\frac{\sin u\lambda_0}{u\lambda_0}\right)^4 du \quad . \quad . \quad . \quad . \quad (63).$$

Now $(\sin u\lambda_0/u\lambda_0)^4$ is always < 1, so that

$$f < \frac{2\mathrm{W}}{\pi}\left(\frac{l}{2b}\right)^4\int_0^\infty \frac{\frac{4}{\mathrm{E}}u^3\cosh u + \frac{u^4\sinh u}{\mu}}{\sinh 2u - 2u}\,du,$$

and f tends to become equal to the right-hand side of the last written inequality if $l/2b$ becomes small, that is, if we make our supports close up.

The integrals $\int_0^\infty \frac{u^3\cosh u\,du}{\sinh 2u - 2u}$ and $\int_0^\infty \frac{u^4\sinh u\,du}{\sinh 2u - 2u}$ when calculated by quadratures come out to be equal to 7·22 and 24·82 respectively.

We have therefore $$f < \frac{2\mathrm{W}}{\pi}\left(\frac{l}{2b}\right)^4\left(\frac{28{\cdot}9}{\mathrm{E}} + \frac{24{\cdot}8}{\mu}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

Now if f_0 be the Euler-Bernoulli deflection, that is, the deflection calculated in the usual way by taking the curvature proportional to the bending moment and fixing, so that the elastic line is horizontal at the origin,

$$f_0 = \frac{\mathrm{W}l^3}{4\mathrm{E}b^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Comparing (64) and (65) we see that the true deflection will certainly be less than the Euler-Bernoulli deflection if $l\left(28{\cdot}9 + \frac{E}{\mu}24{\cdot}8\right) < 2\pi b$; or, $l < {\cdot}069\,b$, if for purposes of numerical calculation we suppose uni-constant isotropy and therefore $E = 5\mu/2$.

So that if l be less than about $\frac{1}{28}$th of the height of the beam, the correction to be applied to the Euler-Bernoulli deflection becomes negative. The critical point where, as we shorten the span, the correction passes from additive to subtractive corresponds to l slightly, but only very slightly, greater than $\cdot069\,b$, as in the neighbourhood of this value λ_0 is quite sufficiently small to make $(\sin u\lambda_0/u\lambda_0)^4 = 1$, a fair approximation for all the most important part of the range of integration of the integral in (63).

We see therefore that when we have a beam loaded in this way, with a section of symmetry constrained to remain plane, the deflection at the centre, for all spans greater than $\frac{1}{28}$th of the height, is larger than the one indicated by the Euler-Bernoulli theory. In the limit when the span is made very large, this additive correction is found to be of the same form as that given by DE SAINT-VENANT for a cantilever under special conditions of end fixing, but the coefficient is different, the correction being just under $\frac{3}{4}$ths of DE SAINT-VENANT'S value. For spans smaller than $\frac{1}{28}$th of the height the correction is negative.

§ 14. *The Doubly-supported Beam under Central Load. Expressions for the Strains and Stresses when we remove the Supports to the Two Extremities.*

Going back to the general expressions for U, V, P, Q, S given in § 10, if we have a beam as in § 13, but we make the two supports coincide with $x = \pm a$, we have

$$\alpha_0 = \beta_0 = -\frac{W}{2a}, \quad \alpha_n = -\frac{W}{a}, \quad \beta_n = -(-1)^n\frac{W}{a},$$

with the following values for the displacements and stresses :—

$$\begin{aligned}
U = &-\frac{y}{\mu}\sum_1^\infty \frac{W}{a}\frac{\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a}\sin 2n\pi x/a \sinh 2n\pi y/a \\
&-\frac{y}{\mu}\frac{W}{a}\sum_0^\infty \frac{\cosh(2n+1)\pi b/a}{\sinh(4n+2)\pi b/a - (4n+2)\pi b/a}\sin(2n+1)\pi x/a\cosh(2n+1)\pi y/a \\
&-\frac{W}{a}\sum_0^\infty\frac{a}{(2n+1)\pi}\left(\frac{\left[\begin{array}{l}\frac{1}{\lambda'+\mu}\cosh(2n+1)\pi b/a \\ -\frac{1}{\mu}\frac{(2n+1)\pi b}{a}\sinh(2n+1)\pi b/a\end{array}\right]}{\sinh(4n+2)\pi b/a - (4n+2)\pi b/a}\right)\sin(2n+1)\pi x/a\sinh(2n+1)\pi y/a \\
&-\frac{W}{a}\sum_1^\infty\frac{a}{2n\pi}\frac{\left(\frac{1}{\lambda'+\mu}\sinh 2n\pi b/a - \frac{1}{\mu}\frac{2n\pi b}{a}\cosh 2n\pi b/a\right)}{\sinh 4n\pi b/a + 4n\pi b/a}\sin 2n\pi x/a\cosh 2n\pi y/a \\
&-\frac{3Wa}{\pi}xy\sum_0\frac{1}{(2n+1)^2} - \frac{\eta W x}{2Ea} \quad \ldots \ldots \quad (66).
\end{aligned}$$

$$V = \frac{yW}{\mu a} \sum_1^\infty \frac{\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \cosh 2n\pi y/a$$

$$+ \frac{yW}{\mu a} \sum_0^\infty \frac{\cosh (2n+1)\pi b/a}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos (2n+1)\pi x/a \sinh (2n+1)\pi y/a$$

$$- \frac{W}{a} \sum_1^\infty \frac{a}{2n\pi} \frac{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\sinh 2n\pi b/a + \frac{1}{\mu}\frac{2n\pi b}{a}\cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \sinh 2n\pi y/a$$

$$- \frac{W}{a} \sum_1^\infty \frac{a}{(2n+1)\pi} \frac{\left[\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\cosh \widehat{2n+1}\pi b/a + \frac{1}{\mu}\frac{\widehat{2n+1}\pi b}{a}\sinh \widehat{2n+1}\pi b/a\right]}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos (2n+1)\pi x/a \cosh (2n+1)\pi y/a$$

$$+ \tfrac{3}{2}\frac{Wa}{Eb^3\pi^2}(x^2 - \eta y^2) \sum_0^\infty \frac{1}{(2n+1)^2} - \frac{Wy}{Ea} + B \quad . \quad . \quad . \quad . \quad (67),$$

where B is a constant depending on the origin from which the displacement is to be measured.

$$P = -\frac{3yWa}{b^3\pi^2} \sum_0^\infty \frac{1}{(2n+1)^2}$$

$$- \sum_1^\infty \frac{2W}{a} \frac{\sinh 2n\pi b/a - (2n\pi b/a)\cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \cosh 2n\pi y/a$$

$$- \sum_1^\infty \frac{2W}{a} \frac{(2n\pi y/a)\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \sinh 2n\pi y/a$$

$$- \sum_0^\infty \frac{2W}{a} \frac{\cosh \widehat{2n+1}\,\pi b/a - (\widehat{2n+1}\pi b/a)\sinh \widehat{2n+1}\pi b/a}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos \widehat{2n+1}\pi x/a \sinh \widehat{2n+1}\pi y/a$$

$$- \sum_0^\infty \frac{2W}{a} \frac{(\widehat{2n+1}\pi y/a)\cosh \widehat{2n+1}\pi b/a}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos \widehat{2n+1}\pi x/a \cosh \widehat{2n+1}\pi y/a \quad . \quad . \quad . \quad (68).$$

$$Q = -\frac{W}{2a} - \sum_1^\infty \frac{2W}{a} \frac{\sinh 2n\pi b/a + (2n\pi b/a)\cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \cosh 2n\pi y/a$$

$$+ \sum_1^\infty \frac{2W}{a} \frac{(2n\pi y/a)\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \sinh 2n\pi y/a$$

$$- \sum_0^\infty \frac{2W}{a} \frac{\cosh \widehat{2n+1}\pi b/a + (\widehat{2n+1}\pi b/a)\sinh \widehat{2n+1}\pi b/a}{\sinh \widehat{4n+2}\pi b/a - \widehat{4n+2}\pi b/a} \cos \widehat{2n+1}\pi x/a \sinh \widehat{2n+1}\pi y/a$$

$$+ \sum_0^\infty \frac{2W}{a} \frac{(\widehat{2n+1}\pi y/a)\cosh \widehat{2n+1}\pi b/a}{\sinh \widehat{4n+2}\pi b/a - \widehat{4n+2}\pi b/a} \cos \widehat{2n+1}\pi x/a \cosh \widehat{2n+1}\pi y/a \quad . \quad . \quad . \quad . \quad (69).$$

$$S = \sum_1^\infty \frac{2W}{a} \frac{(2n\pi b/a)\cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \sin 2n\pi x/a \sinh 2n\pi y/a$$

$$- \sum_1^\infty \frac{2W}{a} \frac{(2n\pi y/a)\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \sin 2n\pi x/a \cosh 2n\pi y/a$$

$$+ \sum_0^\infty \frac{2W}{a} \frac{(\overline{2n+1}\pi b/a)\sinh \overline{2n+1}\pi b/a}{\sinh \overline{4n+2}\pi b/a - \overline{4n+2}\pi b/a} \sin \overline{2n+1}\pi x/a \cosh \overline{2n+1}\pi y/a$$

$$- \sum_0^\infty \frac{2W}{a} \frac{(\overline{2n+1}\pi y/a)\cosh \overline{2n+1}\pi b/a}{\sinh \overline{4n+2}\pi b/a - \overline{4n+2}\pi b/a} \sin \overline{2n+1}\pi x/a \sinh \overline{2n+1}\pi y/a \quad . \quad . \quad . \quad . \quad (70).$$

§ 15. *Definite Integrals to which the expressions of the last Section tend when we make "a" very large.*

If we make a very large, the Σ's in the preceding expressions will become integrals in the limit. It will be found, however, that certain terms in the last found values of U, V, P, Q, S become infinite when 0 is substituted for b/a. In these cases the sum may not be directly transformed into an integral. The reason why this occurs is that, if a be made infinite, an infinite bending moment $\frac{1}{2}Wa$ is introduced at the centre of the beam. It is this moment which produces the parts of the displacements and stresses that become infinite when a is infinite. If, however, we apply at the two ends pure couples $-\frac{1}{2}Wa$, we get rid of this infinite moment, and we have only the terms due to the local effect, which produces only finite stresses at a finite distance from the origin.

Thus, if in U we add $\frac{yW}{a}\sum_0^\infty \frac{3}{4}\frac{1}{\mu}\frac{xa^2}{b^3(2n+1)^2\pi^2}$ to the second Σ and $\frac{W}{a}\sum_0^\infty \frac{3}{4}\frac{1}{\lambda'+\mu}\frac{xya^2}{b^3(2n+1)^2\pi^2}$ to the third Σ, these Σ's remain finite even when we make $a = \infty$. We have, however, to introduce negative terms to balance those that have been added. Remembering that $\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right) = \frac{4}{E}$ and $\sum_0^\infty \frac{1}{(2n+1)^2} = \pi^2/8$, we see that the part of the series in U which becomes infinite, is $-\frac{3}{8}\frac{xyWa}{Eb^3}$, which, added to the other infinite term in the last line of (66), gives for the infinite part of U:

$$U_0 = -\frac{3}{4}\frac{xyWa}{Eb^3}.$$

Similarly with V. The terms which have to be added to the second and fourth Σ's to make them finite in the limit are

$$-\frac{3y^2W}{4\mu a}\sum_0^\infty \frac{a^2}{(2n+1)^2\pi^2b^3},$$

$$+\frac{W}{a}\sum_0^\infty \frac{a^4}{(2n+1)^4\pi^4b^3}\frac{3}{4}\left[\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left(1+\frac{3}{10}\frac{(2n+1)^2\pi^2b^2}{a^2}\right)+\frac{1}{\mu}\frac{(2n+1)^2\pi^2b^2}{a^2}\right]$$

$$+\frac{W}{a}\sum_0^\infty \frac{a^2}{(2n+1)^2\pi^2b^3}\frac{3}{8}(y^2-x^2)\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right),$$

respectively. The first sum in the second of these expressions does not contain the variables, and therefore may be supposed taken from the constant B. The other terms, added to the first term of the last line of (67), will give for the infinite part of V

$$V_0 = \tfrac{3}{8}\frac{Wa}{Eb^3}(x^2 - \eta y^2).$$

Proceeding to deal in the same way with the stresses, we find that to ensure finiteness in the limit we must add:

(a) to the third and fourth series in P: $\frac{2W}{a}\sum_0^\infty \frac{3y}{4b^3}\frac{a^2}{(2n+1)^2\pi^2}$ in each case;

(b) to the third and fourth series in Q: $\frac{2W}{a}\sum_0^\infty \frac{3y}{4b^3}\frac{a^2}{(2n+1)^2\pi^2}$ and $-\frac{2W}{a}\sum_0^\infty \frac{3y}{4b^3}\frac{a^2}{(2n+1)^2\pi^2}$ respectively; the infinite part of P is then

$$P_0 = -\tfrac{3}{4}\frac{Wa}{b^3}y,$$

Q and S having no infinite parts.

If we leave out of account the parts U_0, V_0, P_0, which belong to a couple $Wa/2$ and which can be destroyed by introducing an equal and opposite couple, we find that, when a is made infinite, the displacements and stresses tend to the following limiting values:

$$\left.\begin{aligned}
U = &-\frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \left(\frac{\cosh u}{\sinh 2u - 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b} - \frac{3x}{4bu^2}\right)du \\
&-\frac{W}{2\pi}\int_0^\infty \left\{\frac{\frac{1}{\lambda'+\mu}\cosh u - \frac{1}{\mu}u\sinh u}{\sinh 2u - 2u}\,\frac{1}{u}\sin\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3xy\left(\frac{1}{\lambda'+\mu}\right)}{4b^2u^2}\right\}du \\
&-\frac{W}{2\pi}\int_0^\infty \left\{\frac{\frac{1}{\lambda'+\mu}\sinh u - \frac{1}{\mu}u\cosh u}{\sinh 2u + 2u}\right\}\frac{1}{u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
V = &\ \frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \left\{\frac{\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3y}{4bu^2}\right\}du \\
&-\frac{W}{2\pi}\int_0^\infty \frac{\left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sinh u + \frac{1}{\mu}u\cosh u\right\}}{\sinh 2u + 2u}\,\frac{1}{u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{W}{2\pi}\int_0^\infty \left\{\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u + \frac{1}{\mu}u\sinh u}{\sinh 2u - 2u}\,\frac{1}{u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\right. \\
&\qquad\left. -\tfrac{3}{4}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{1}{u^4} - \tfrac{3}{40}\left(\frac{3}{\lambda'+\mu}+\frac{13}{\mu}\right)\frac{1}{u^2} - \tfrac{3}{8}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{y^2-x^2}{b^2u^2}\right\}du \\
&+\text{ an arbitrary constant } B'
\end{aligned}\right\} \quad (71).$$

$$
\left.\begin{aligned}
P = & -\frac{W}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
& -\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& -\frac{W}{\pi b}\int_0^\infty \left\{\frac{\cosh u - u\sinh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3}{4}\frac{y}{bu^2}\right\}du \\
& -\frac{Wy}{\pi b^2}\int_0^\infty \left\{\frac{u\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b} - \frac{3}{4u^2}\right\}du \\
Q = & -\frac{W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
& +\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& -\frac{W}{\pi b}\int_0^\infty \left\{\frac{\cosh u + u\sinh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3}{4}\frac{y}{bu^2}\right\}du \\
& +\frac{Wy}{\pi b^2}\int_0^\infty \left\{\frac{u\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b} - \frac{3}{4u^2}\right\}du \\
S = & \frac{W}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& -\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
& +\frac{W}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
& -\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du
\end{aligned}\right\} \quad . \quad . \quad (72).
$$

§ 16. *Consideration of the Stresses in the Neighbourhood of the Point where the Concentrated Load is Applied.*

The integrals in the expressions (71) and (72) are finite, one-valued and continuous at every point (x, y) inside the beam, such that y is numerically less than b by a finite quantity. For in this case, for large values of u, the integrand is comparable with $e^{-u(b-|y|)}$, where $|y|$ stands for the numerical value of y. If, however, $|y| = b$, or the point in question lies on the edges of the beam, the integrals are no longer necessarily convergent. In this case the expressions (71) and (72) have to be transformed.

Let us start with the stresses P, Q, S, as in their case the transformation is somewhat simpler. Further, let us consider instead of P and Q the somewhat more compactly expressed quantities

$$\left.\begin{aligned}
P + Q &= -\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&\quad - \frac{2W}{\pi b}\int_0^\infty\left\{\frac{\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3y}{4u^2 b}\right\}du \\
P - Q &= \frac{2W}{\pi b}\int_0^\infty \frac{u\cosh u\cosh\frac{uy}{b} - \frac{uy}{b}\sinh u\sinh\frac{uy}{b}}{\sinh 2u + 2u}\cos\frac{ux}{b}\,du \\
&\quad + \frac{2W}{\pi b}\int_0^\infty\left\{\frac{u\sinh u\sinh\frac{uy}{b} - \frac{uy}{b}\cosh u\cosh\frac{uy}{b}}{\sinh 2u - 2u}\cos\frac{ux}{b} + \frac{3}{4}\frac{y}{bu^2}\right\}du
\end{aligned}\right\} \quad (73).$$

$P - Q$ and S give the lines of principal stress and the principal stress-difference, $P + Q$ gives the compression at the point considered.

If in the values (73) and in S we write $y = b - y'$ so that we are referring our co-ordinates x, y' to the point C (fig. i.) where the concentrated load is applied, as origin, we find, on re-arranging the terms,

$$\begin{aligned}
P + Q = &-\frac{2W}{\pi b}\int_0^\infty e^{-\frac{uy'}{b}}\cos\frac{ux}{b}\,du \\
&- \frac{2W}{\pi b}\int_0^\infty\left\{\left[\frac{u}{\sinh 2u - 2u} - \frac{u}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{4u^2}\right\}du \\
&+ \frac{2W}{\pi b}\int_0^\infty\left\{\frac{1}{2}\left[\frac{1 + 2u + e^{-2u}}{\sinh 2u - 2u} - \frac{1 + 2u - e^{-2u}}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{4u^2}\frac{y'}{b}\right\}du.
\end{aligned}$$

$$\begin{aligned}
P - Q = &\frac{2Wy'}{\pi b^2}\int_0^\infty ue^{-\frac{uy'}{b}}\cos\frac{ux}{b}\,du \\
&- \frac{2W}{\pi b}\int_0^\infty\left\{\left[\frac{u}{\sinh 2u - 2u} - \frac{u}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{4u^2}\right\}du \\
&+ \frac{2Wy'}{\pi b^2}\int_0^\infty\left\{\frac{u}{2}\left[\frac{1 + 2u + e^{-2u}}{\sinh 2u - 2u} - \frac{1 + 2u - e^{-2u}}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{4u^2}\right\}du \\
&- \frac{2Wy'}{\pi b^2}\int_0^\infty\left[\frac{u^2}{\sinh 2u - 2u} - \frac{u^2}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\sinh\frac{uy'}{b}\,du.
\end{aligned}$$

$$\begin{aligned}
S = &\frac{Wy'}{\pi b^2}\int_0^\infty ue^{-\frac{uy'}{b}}\sin\frac{ux}{b}\,du \\
&+ \frac{W}{\pi b}\int_0^\infty\left[\frac{u}{\sinh 2u - 2u} - \frac{u}{\sinh 2u + 2u}\right]\sin\frac{ux}{b}\sinh\frac{uy'}{b}\,du \\
&- \frac{Wy'}{\pi b^2}\int_0^\infty\frac{u}{2}\left[\frac{1 + 2u + e^{-2u}}{\sinh 2u - 2u} - \frac{1 + 2u - e^{-2u}}{\sinh 2u + 2u}\right]\sin\frac{ux}{b}\sinh\frac{uy'}{b}\,du \\
&+ \frac{Wy'}{\pi b^2}\int_0^\infty\left[\frac{u^2}{\sinh 2u - 2u} - \frac{u^2}{\sinh 2u + 2u}\right]\sin\frac{ux}{b}\cosh\frac{uy'}{b}\,du.
\end{aligned}$$

The leading integrals in each case can be evaluated.

If we write $x = r' \sin \phi'$ $y' = r' \cos \phi'$, so that r' is the distance of the point considered from the point of application of the concentrated load and ϕ' is the angle which r' makes with the vertical, then :

$$\int_0^\infty e^{-\frac{uy'}{b}} \cos \frac{ux}{b}\, du = \frac{b \cos \phi'}{r'}$$

$$\int_0^\infty u e^{-\frac{uy'}{b}} \cos \frac{ux}{b}\, du = \frac{b^2 \cos 2\phi'}{r'^2}$$

$$\int_0^\infty u e^{-\frac{uy'}{b}} \sin \frac{ux}{b}\, du = \frac{b^2 \sin 2\phi'}{r'^2}$$

and we have

$$P + Q = -\frac{2W}{\pi r'} \cos \phi' - \frac{8W}{\pi b}\int_0^\infty \left\{\frac{u^2}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \cosh \frac{uy'}{b} - \frac{3}{16u^2}\right\} du$$
$$+ \frac{8W}{\pi b}\int_0^\infty \left\{\frac{u^2 + \frac{u}{2} + \frac{1}{8} - \frac{1}{8} e^{-4u}}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \sinh \frac{uy'}{b} - \frac{3}{16u^2} \frac{y'}{b}\right\} du \quad . \quad . \quad . \quad (74).$$

$$P - Q = \frac{2Wy'}{\pi r'^2} \cos 2\phi' - \frac{8W}{\pi b}\int_0^\infty \left\{\frac{u^2}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \cosh \frac{uy'}{b} - \frac{3}{16u^2}\right\} du$$
$$+ \frac{8Wy'}{\pi b^2}\int_0^\infty \left\{\frac{u^3 + \frac{u^2}{2} + \frac{u}{8} - \frac{u}{8} e^{-4u}}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \cosh \frac{uy'}{b} - \frac{3}{16u^2}\right\} du$$
$$- \frac{8Wy'}{\pi b^2}\int_0^\infty \frac{u^3}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \sinh \frac{uy'}{b}\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

$$S = \frac{Wy'}{\pi r'^2} \sin 2\phi' + \frac{4W}{\pi b}\int_0^\infty \frac{u^2}{\sinh^2 2u - 4u^2} \sin \frac{ux}{b} \sinh \frac{uy'}{b}\, du$$
$$- \frac{4Wy'}{\pi b^2}\int_0^\infty \frac{u^3 + \frac{u^2}{2} + \frac{u}{8} - \frac{u}{8} e^{-4u}}{\sinh^2 2u - 4u^2} \sin \frac{ux}{b} \sinh \frac{uy'}{b}\, du$$
$$+ \frac{4Wy'}{\pi b^2}\int_0^\infty \frac{u^3}{\sinh^2 2u - 4u^2} \sin \frac{ux}{b} \cosh \frac{uy'}{b}\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

The expressions for the stresses therefore consist of two parts, namely, the integrated parts

$$\left.\begin{aligned} P_1 &= -\frac{W}{\pi r'} \cos \phi' + \frac{Wy'}{\pi r'^2} \cos 2\phi' = -\frac{2W}{\pi} \frac{x^2 y'}{r'^4} \\ Q_1 &= -\frac{W}{\pi r'} \cos \phi' - \frac{Wy'}{\pi r'^2} \cos 2\phi' = -\frac{2W}{\pi} \frac{y'^3}{r'^4} \\ S_1 &= \frac{Wy'}{\pi r'^2} \sin 2\phi' \qquad\qquad\quad = \frac{2W}{\pi} \frac{xy'^2}{r'^4} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (77),$$

and the parts still in the form of integrals, which we may call P_2, Q_2, S_2.

P_1, Q_1, S_1 agree with the expressions found by FLAMANT ('Comptes Rendus,' vol. 114, pp. 1465–1468) and confirmed by BOUSSINESQ ('Comptes Rendus,' vol. 114, pp. 1510–1516) for the stresses in an infinite solid due to a line of load W per unit length, in which case the problem is reduced to two dimensions. They correspond, therefore, to the stresses that would be induced in the beam by the concentrated load if the height $2b$ were made infinite.

The stresses P_2, Q_2, S_2 are regular functions of x and y throughout the beam. They nowhere become discontinuous or infinite, and they tend to zero as b is made large. They represent the correction that we have to apply to FLAMANT'S and BOUSSINESQ'S result as a consequence of the finite height of the beam.

BOUSSINESQ, in the paper quoted above, has made an attempt to obtain such a correction, by finding the stresses given by (77) over the lower edge of the beam, superimposing an equal and opposite system to annul these, and calculating the strains due to this last system as if the *top* boundary of the beam were removed to infinity. This corrective system, as he calls it, will now introduce extra stresses over the top of the beam. To get rid of these a corrective system of the second order is superimposed, and we may go on indefinitely in this way. The complexity of the expressions increases enormously for each system we add, and, on finding the approximation so slowly convergent that the terms of the second order were practically as important as those of the first, BOUSSINESQ threw up the method in despair, and fell back upon an empirical assumption, given by Sir GEORGE STOKES in a supplement to a paper by CARUS WILSON ('Phil. Mag.,' Series V., vol. 32, pp. 500–503), namely, that the stress system introduced by the finiteness of the height of the beam was such as to annul the stresses due to (77) at the lower boundary, and varied linearly along the vertical, giving zero stress over the upper boundary. The functions P_2, Q_2, S_2 of the present article solve the problem exactly.

§ 17. *Expansion in Integral Powers about the Point of Discontinuous Loading.*

In the integrals for P_2, Q_2, S_2 we may expand the quantities $\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{ux}{b} \times \left.\begin{matrix}\cosh\\ \sinh\end{matrix}\right\}\frac{uy'}{b}$ in series as follows:—

$$\left.\begin{aligned}
\sin\frac{ux}{b}\sinh\frac{uy'}{b} &= \sum_1^\infty \left(\frac{ur'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}\\
\sin\frac{ux}{b}\cosh\frac{uy'}{b} &= \sum_0^\infty \left(\frac{ur'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)}\\
\cos\frac{ux}{b}\cosh\frac{uy'}{b} &= \sum_0^\infty \left(\frac{ur'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}\\
\cos\frac{ux}{b}\sinh\frac{uy'}{b} &= \sum_0^\infty \left(\frac{ur'}{b}\right)^{2\nu+1}\frac{\cos\overline{2\nu+1}\,\phi'}{(2\nu+1)!}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (78),$$

ν being an integer. Now when these values are substituted in (77) and similar formulæ, we may distribute the integral sign among the terms of the series, provided that both the original and the resulting series are absolutely and uniformly convergent. This is easily seen to hold good for the series (78), and it will be shown later, in § 18, to be true of the resulting series, providing the points considered lie inside a certain circle of convergence.

Assuming for the moment this result, we obtain from (77)

$$\left.\begin{aligned} P_2 &= -\frac{4W}{\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu} H_{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!} - \frac{4W}{\pi b}\sum_0^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu \frac{\cos \nu\phi'}{\nu} H_\nu \\ &\quad + \frac{4Wy'}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu H_{\nu+1}\frac{\cos\nu\phi'}{\nu!} \\ Q_2 &= \frac{4W}{\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1} H_{2\nu+1}\frac{\cos \widehat{2\nu+1}\phi'}{(2\nu+1)!} - \frac{4Wy'}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu H_{\nu+1}\frac{\cos\nu\phi'}{\nu!} \\ S_2 &= \frac{4W}{\pi b}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu} H_{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!} - \frac{4Wy'}{\pi b^2}\sum_1^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu H_{\nu+1}\frac{\sin\nu\phi'}{\nu!} \end{aligned}\right\} \quad . \quad (79),$$

where

$$\left.\begin{aligned} H_0 &= \int_0^\infty\left(\frac{u^2}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right)du \\ H_1 &= \int_0^\infty\left(\frac{u^3+\frac{1}{2}u^2+\frac{1}{8}u-\frac{1}{8}u\,e^{-4u}}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right)du \\ H_{2\nu} &= \int_0^\infty \frac{u^{2\nu+2}}{\sinh^2 2u - 4u^2}\,du \\ H_{2\nu+1} &= \int_0^\infty\left(\frac{u^{2\nu+3}+\frac{1}{2}u^{2\nu+2}+\frac{1}{8}u^{2\nu+1}-\frac{1}{8}u^{2\nu+1}e^{-4u}}{\sinh^2 2u - 4u^2}\right)du \\ &\qquad (\nu > 0). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (80).$$

§ 18. *Convergency of the Series of the last Section.*

In order to justify the distribution of the integral sign over the separate terms of the series (78), we have to show that the series (79) are absolutely and uniformly convergent.

Now the series are absolutely and uniformly convergent provided that the series $\sum_0^\infty\left(\frac{r'}{b}\right)^\nu \frac{H_\nu}{\nu!}$ is absolutely and uniformly convergent. The convergency ratio of this latter series $= \underset{\nu=\infty}{L}\ \frac{r'}{b}\frac{H_{\nu+1}}{H_\nu}\frac{1}{\nu+1}$.

Now, in order to find the approximate value of H_ν when ν is large, let us consider the integral

$$I_n = \int_0^\infty \frac{u^n}{\sinh^2 2u - 4u^2}\,du$$

write $u = av$

$$I_n = a^{n+1}\int_0^\infty \frac{v^n\,dv}{\sinh^2 2av - 4a^2v^2} = a^{n+1}\int_0^\infty \frac{4v^n\,e^{-4av}\,dv}{1+e^{-8av}-(2+16a^2v^2)\,e^{-4av}}.$$

Now let a be chosen so large that for all values of $v > \omega$, where ω is numerically less than unity,

$$(2 + 16a^2v^2)\, e^{-4av} - e^{-8av} < \epsilon,$$

where ϵ is a small, finite, assigned quantity.

We then find

$$\mathrm{I}_n = a^{n+1} \left\{ \int_0^\omega \frac{2v^n\, dv}{\sinh^2 2av - 4a^2v^2} + \mathrm{U}_n \right\},$$

where U_n lies between $\int_\omega^\infty 4v^n e^{-4av}\, dv$ and $\frac{1}{1+\epsilon}\int_\omega^\infty 4v^n e^{-4av}\, dv$.

Now

$$\int_\omega^\infty v^n e^{-4av}\, dv$$

$$= \frac{n!\, e^{-4a\omega}}{(4a)^{n+1}} \left(1 + \frac{4a\omega}{1!} + \frac{(4a\omega)^2}{2!} + \ldots + \frac{(4a\omega)^n}{n!}\right)$$

$$= \frac{n!\, e^{-4a\omega}}{(4a)^{n+1}} \left(e^{4a\omega} - \left\{\frac{(4a\omega)^{n+1}}{(n+1)!} + \frac{(4a\omega)^{n+2}}{(n+2)!} + \ldots \text{ to } \infty\right\}\right)$$

$$= \frac{n!}{(4a)^{n+1}} \left(1 - e^{-4a\omega}\left\{\frac{(4a\omega)^{n+1}}{(n+1)!} + \ldots \text{ to } \infty\right\}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

Next

$$\sinh^2 2av - 4a^2v^2 > \tfrac{16}{3}a^4v^4.$$

Therefore

$$\int_0^\omega \frac{2v^n\, dv}{\sinh^2 2av - 4a^2v^2} < \int_0^\omega \frac{3}{8}\frac{v^{n-4}}{a^4}\, dv < \frac{3}{8}\frac{\omega^{n-3}}{a^4(n-3)}.$$

Now, ω being > 1, this tends to zero when n is large. Further, by making n sufficiently large, the second term in (81) is negligible compared with the first.

We then find that the most important terms in I_n lie between $\frac{1}{1+\epsilon}\,\frac{n!}{4^n}$ and $\frac{n!}{4^n}$. Hence when n is large we may neglect I_{n-1}, I_{n-2}, &c., compared with I_n.

Now

$$\mathrm{H}_{2\nu} = \mathrm{I}_{2\nu+2},$$

$$\mathrm{H}_{2\nu+1} = \mathrm{I}_{2\nu+3} + \tfrac{1}{2}\mathrm{I}_{2\nu+2} + \tfrac{1}{8}\mathrm{I}'_{2\nu+1},$$

where

$$\mathrm{I}'_{2\nu+1} = \int_0^\infty \frac{u^{2\nu+1}(1 - e^{-4u})}{\sinh^2 2u - 4u^2}\, du < \int_0^\infty \frac{u^{2\nu+1}}{\sinh^2 2u - 4u^2}\, du < \mathrm{I}_{2\nu+1}.$$

Therefore $\mathrm{H}_{2\nu+1} = \mathrm{I}_{2\nu+3}$ if we neglect all but the most important terms. Therefore in the limit $\mathrm{H}_\nu = \mathrm{I}_{\nu+2}$.

$$\text{Convergency ratio} = \underset{\nu=\infty}{\mathrm{L}} \left(\frac{r'}{b}\right) \frac{\mathrm{I}_{\nu+3}}{\mathrm{I}_{\nu+2}}\, \frac{1}{\nu+1}$$

$$= \underset{\nu=\infty}{\mathrm{L}} \left(\frac{r'}{b}\right) \frac{\left(\frac{1}{1+\theta\epsilon}\right)}{\left(\frac{1}{1+\theta'\epsilon}\right)}\, \frac{\nu+3}{4(\nu+1)} = \frac{r'}{4b}\, \frac{1+\theta'\epsilon}{1+\theta\epsilon},$$

where θ, θ' are proper fractions. If we take ϵ small enough, the convergency ratio tends to $\frac{r'}{4b}$.

The series we are dealing with are therefore absolutely and uniformly convergent inside a circle whose centre is the point where the concentrated load is applied and whose radius is twice the height of the beam.

The transformation used in the previous section was therefore justifiable for this region and the expressions (79) are real arithmetical equivalents of the stresses P_2, Q_2, S_2, which have to be superimposed upon FLAMANT and BOUSSINESQ's solutions for an infinite solid when we take into account the height of the beam. The values of the first few coefficients, calculated approximately by quadratures, were found to be as follows: $H_0 = -\cdot2417$, $H_1 = -\cdot0598$, $H_2 = +\cdot2271$, $H_3 = +\cdot3370$.

§ 19. *Transformed Expressions for the Displacements.*

If we take the expressions (71) for U and V, we may treat them exactly as we treated the expressions for P, Q, S. We then obtain, after some rather lengthy reductions, $U = U_1 + U_2$, $V = V_1 + V_2$, where

$$\left.\begin{aligned} U_1 &= \frac{1}{\mu}\frac{Wy'}{2\pi b}\int_0^\infty e^{-\frac{uy'}{b}}\sin\frac{ux}{b}\,du - \frac{W}{2\pi}\frac{1}{\lambda'+\mu}\int_0^\infty \frac{1}{\mu}e^{-\frac{uy'}{b}}\sin\frac{ux}{b}\,du \\ &= \frac{1}{\mu}\frac{Wy'}{2\pi r'}\sin\phi' - \frac{W}{2\pi}\frac{1}{\lambda'+\mu}\phi' \\ V_1 &= -\frac{1}{\mu}\frac{Wy'}{2\pi b}\int_0^\infty e^{-\frac{uy'}{b}}\cos\frac{ux}{b}\,du - \frac{W}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty \frac{\cos\frac{ux}{b}\,e^{-\frac{uy'}{b}} - e^{-u\beta}}{u}\,du + B_1 \\ &= -\frac{1}{\mu}\frac{W}{2\pi}\frac{y'}{r'}\cos\phi' + \frac{W}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\log\left(\frac{r'}{\beta b}\right) + B_1 \end{aligned}\right\} \quad (82).$$

$$\left.\begin{aligned} U_2 = \frac{1}{\mu}\frac{2Wy'}{\pi b}&\int_0^\infty\left[\frac{u^2+\frac{u}{2}+\frac{1}{8}+\frac{1}{8}e^{-4u}}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3x}{16u^2 b}\right]du \\ &- \frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty \frac{u^2}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy'}{b}\,du \\ &- \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty\left[\frac{u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{16}\frac{x}{u^2 b}\right]du \\ &+ \frac{2W}{\pi}\frac{1}{\lambda'+\mu}\int_0^\infty\left[\frac{u+\frac{1}{2}+\frac{1}{8}u^{-1}-\frac{1}{8}u^{-1}e^{-4u}}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{16}\frac{xy'}{u^2 b}\right]du \end{aligned}\right\} \quad (83),$$

$$
\left.\begin{aligned}
V_2 = &-\frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty\left(\frac{u^2}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{16u^2}\right)du \\
&+\frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty\left[\frac{u^2+\frac{1}{2}u+\frac{1}{8}-\frac{1}{8}e^{-4u}}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{16}\frac{y'}{bu^2}\right]du \\
&-\frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty\left[\frac{u+\frac{1}{2}+\frac{1}{8}u^{-1}-\frac{1}{8}u^{-1}e^{-4u}}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy'}{b}\right. \\
&\qquad\qquad\left. - \frac{3}{16u^4} - \frac{3}{20u^2} - \frac{3}{32u^2}\frac{y'^2-x^2}{b^2} + \frac{e^{-u\beta}}{4u}\right]du \\
&+\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\int_0^\infty\left[\frac{u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{16}\frac{y'}{bu^2}\right]du + B_2
\end{aligned}\right\} \quad (83),
$$

where β, B_1, B_2 are arbitrary constants.

The expressions U_1, V_1 agree with those found by BOUSSINESQ in the paper referred to above ('Comptes Rendus,' vol. 114, pp. 1500–1516) for the displacements when b is made infinite. We see that U is indeterminate and V infinite at the point where the concentrated load acts.

Of course such infinite and indeterminate displacements could not occur in nature. With any real material, if it were possible to approximate to a true knife-edge, the infinite stress under the knife-edge would at once either cause the material to break, or else—and this is what must almost always occur in practice—reduce the parts in the immediate neighbourhood of the knife-edge to a plastic condition, so that in this region the equations of elasticity would no longer apply.

Hence for practical applications we have to exclude the actual line of application of the load, $r' = 0$, and a very thin cylinder surrounding it. If we do this, then all our results will be valid for points whose distance from the knife-edge is at all large compared with the radius of this thin cylinder. A notable point about the results (82) is that U_1 is independent of r' and depends only upon the angular co-ordinate of the point considered with regard to the knife-edge as origin. Hence all points lying on a plane through this knife-edge receive the same horizontal displacement

The parts U_2, V_2, of the displacements are finite, one-valued, and continuous throughout the beam and over the edges. They can be, like the stresses P_2, Q_2, S_2, expanded in series of powers of r', which are absolutely and uniformly convergent within a circle of radius $4b$.

These expansions are easily seen to be the following :

$$
\left.\begin{aligned}
U_2 = &-\frac{2Wy'}{\mu\pi b}\sum_1^\infty(-1)^\nu H_\nu\left(\frac{r'}{b}\right)^\nu\frac{\sin\nu\phi'}{\nu!} - \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu} \\
&+\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu-1} \\
V_2 = &-\frac{2Wy'}{\mu\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^\nu(-1)^\nu H_\nu\frac{\cos\nu\phi'}{\nu!} - \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left[H_{-1}+\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}H_{2\nu-1}\right] \\
&+\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+\ )!}H_{2\nu}
\end{aligned}\right\} \quad (84).
$$

where H_{-1} is an arbitrary constant, and the other H's have the same meaning as before.

These equations represent the effect of the finite height of the beam upon the displacements. If in them we put $y' = 0$, $\phi' = \pi/2$, we have the alteration in the displacements over the upper surface due to the finite thickness. This gives us, retaining only the leading terms,

$$(V_2)_{y'=0} = \frac{2W}{\pi}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\frac{H_1}{2\,!}\frac{x^2}{b^2} = -\frac{4W}{\pi E}(\cdot 0598)\frac{x^2}{b^2},$$

$$(U_2)_{y'=0} = -\frac{2W}{\pi}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)H_0\,{}_b = \frac{1\cdot 9336 Wx}{\pi E b},$$

giving a downward curvature at the point of discontinuous load equal to $\frac{\cdot 478 W}{\pi E b^2}$ and a horizontal stretch $\frac{1\cdot 934 W}{\pi E b}$. The effect of the finite thickness appears therefore to be to stiffen the beam and to decrease its curvature under the load.

§ 20. *Expansions about Other Points. Expansion about the Origin.*

The expressions (71) and (72) are capable of being expanded in many other ways. Considering only expansions in powers of the radius vector from a given point, we may write in U, V, P, Q, S : $x = X + \rho \sin \theta$, $y = Y + \rho \cos \theta$, and we shall obtain an expansion which is valid for all points which are contained between $y = \pm b$, and which lie inside a circle with centre (X, Y) passing through the point $(0, +b)$. The coefficients of $\rho^n \cos n\theta$, $\rho^n \sin n\theta$, &c., will be integrals containing X, Y.

The only expansions worth considering are those about the origin and those about the point $(0, -b)$, which is vertically below the load.

The expansions about the origin are deduced immediately from (71) and (72). They are

$$\left.\begin{aligned} U &= -\frac{1}{\mu}\frac{Wy}{2\pi b}\sum^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin \nu\phi}{\nu\,!}F_{\nu} - \frac{W}{2\pi}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin \nu\phi}{\nu\,!}\left(\frac{1}{\lambda' + \mu}F_{\nu-1} - \frac{1}{\mu}G_{\nu}\right) \\ V &= \frac{1}{\mu}\frac{Wy}{2\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos \nu\phi}{\nu\,!}F_{\nu} - \frac{W}{2\pi}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos \nu\phi}{\nu\,!}\left\{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)F_{\nu-1} + \frac{1}{\mu}G_{\nu}\right\} + G_0 \end{aligned}\right\} \quad (85),$$

$$\left.\begin{aligned} P &= -\frac{W}{\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos \nu\phi}{\nu\,!}(F_{\nu} - G_{\nu+1}) - \frac{Wy}{\pi b^2}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos \nu\phi}{\nu\,!}F_{\nu+1} \\ Q &= -\frac{W}{\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos \nu\phi}{\nu\,!}(F_{\nu} + G_{\nu+1}) + \frac{Wy}{\pi b^2}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos \nu\phi}{\nu\,!}F_{\nu+1} \\ S &= \frac{W}{\pi b}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin \nu\phi}{\nu\,!}G_{\nu+1} - \frac{Wy}{\pi b^2}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin \nu\phi}{\nu\,!}F_{\nu+1} \end{aligned}\right\} \quad . \quad . \quad (86),$$

where $y = r\cos\phi$, $x = r\sin\phi$,

$$F_1 = \int_0^\infty \left(\frac{u\cosh u}{\sinh 2u - 2u} - \frac{3}{4u^2}\right) du,$$

$$F_{2\nu+1} = \int_0^\infty \frac{u^{2\nu+1}\cosh u}{\sinh 2u - 2u}\, du \; (\nu > 0),$$

$$F_{2\nu} = \int_0^\infty \frac{u^{2\nu}\sinh u}{\sinh 2u + 2u}\, du \; (\nu \geqq 0),$$

$$G_{2\nu+1} = \int_0^\infty \frac{u^{2\nu+1}\cosh u}{\sinh 2u + 2u}\, du \; (\nu \geqq 0),$$

$$G_{2\nu} = \int_0^\infty \frac{u^{2\nu}\sinh u}{\sinh 2u - 2u}\, du \; (\nu > 0),$$

G_0 = a constant to be adjusted from the fixing conditions. The series in (85) and (86) are absolutely and uniformly convergent inside a circle centre the origin and radius b.

The first few coefficients are given by

$$F_0 = \cdot 527 \qquad G_1 = \cdot 918$$
$$F_1 = \cdot 438 \qquad G_2 = 2\cdot 818$$
$$F_2 = 1\cdot 740 \qquad G_3 = 5\cdot 750$$
$$F_3 = 7\cdot 224 \qquad G_4 = 24\cdot 824,$$

where the integrals have been obtained approximately by quadratures.

Retaining in the expressions (85), (86) only the most important terms, we find for the displacements of points on the x-axis: $U_{y=0} = \frac{W}{2\pi}\frac{x}{b}\left(\frac{1}{\mu}\cdot 918 - \frac{1}{\lambda' + \mu}\cdot 527\right)$, which is positive with x.

We have therefore a horizontal stretch equal to $\frac{W}{2\pi b}\left(\frac{1\cdot 444}{\mu} - \frac{2\cdot 108}{E}\right)$.

For uni-constant isotropy $E = 5\mu/2$, and the stretch is $\frac{W}{2bE}\left(\frac{1\cdot 503}{\pi}\right)$, or about one half the stretch due to the load W acting horizontally along the length of the beam, so as to produce a tension $W/2b$.

Similarly $V_{y=0} = G_0 + \frac{W}{2\pi}\frac{x^2}{b^2}\frac{1}{2!}\left(\frac{4F_1}{E} + \frac{G_2}{\mu}\right)$; this gives a curvature upwards equal to $\frac{W}{2\pi b^2}\left(\frac{1\cdot 753}{E} + \frac{2\cdot 818}{\mu}\right)$, *i.e.*, to the curvature that would be produced by a pure couple $\frac{Wb}{3\pi}\left(1\cdot 753 + \frac{E}{\mu}2\cdot 818\right)$, or (putting $E = 5\mu/2$) by a couple $Wb \times (\cdot 5622)$.

The stresses at points along the x-axis are

$$P_{y=0} = -\frac{W}{\pi b}\left[(F_0 - G_1) - \frac{x^2}{b^2}\frac{1}{2!}(F_2 - G_3)\right]$$

$$= \frac{W}{\pi b}\left[\cdot 391 - \frac{x^2}{b^2} 2\cdot 005\right]$$

$$Q_{y=0} = -\frac{W}{\pi b}\left[1\cdot 444 - \frac{x^2}{b^2} 3\cdot 745\right];$$

we have therefore at the origin a horizontal tension and vertical pressure. These vanish when $x = \pm \cdot 195b$ and $x = \pm \cdot 386b$ respectively, assuming that for these values of x the first two terms are a sufficient approximation, which is certainly true for $x = \cdot 195b$, but only roughly true for $x = \cdot 386b$, as it amounts to neglecting terms of order about $\frac{1}{7}$ compared with 1·44. It will, however, be sufficient for a rough estimate.

The actual stresses at the origin are :—

$P = \frac{W}{2b}(\cdot 249)$, or about $\frac{1}{4}$ of the tension due to W acting along the horizontal,

$Q = -\frac{W}{2b}(\cdot 920)$, or about $\frac{9}{10}$ths of the pressure due to W acting along the horizontal.

If we had used the expressions P_1, Q_1 which hold for an infinite solid, we should find, at the origin, $P = 0$, $Q = -\frac{2W}{\pi b} = -\frac{W}{2b}(1\cdot 273)$.

If we correct the last by STOKES' empirical rule, we have to add $-\frac{1}{2}$ [0 + (stress at bottom of beam as given by the formula for an infinite solid)].

This will give $Q = -\frac{3W}{2\pi b} = -\frac{W}{2b}(\cdot 955)$. The error in the vertical stress, calculated from this amended formula, is therefore only (·035) $W/2b$, or only about $3\frac{1}{2}$ per cent.

With regard to the correction for the horizontal tension, BOUSSINESQ finds, for a span $2l$ and depth $2b$,

$$P = \frac{W}{2b}\left[\frac{4}{\pi} - \frac{3y'}{\pi b} + \frac{3(y'-b)}{2b^2}l\right],$$

where y' is measured from the point $(0, b)$ as before.

The terms $\frac{3W}{4b^3}(y'-b)\,l$ correspond to the bending moment which we have removed.

We have left therefore $P = \frac{W}{2b}\left[\frac{4}{\pi} - \frac{3y'}{\pi b}\right]$, so that, at the origin, when $y' = b$, $P = \frac{W}{2b}\frac{1}{\pi} = \frac{W}{2b}(\cdot 318)$, and this gives a tension which is greater than the actual one by only (·069) $W/2b$.

§ 21. *Expansions about the Point* $(0, -b)$.

It appears of some interest to give the values of the displacements and stresses about the point $(0, -b)$, that is, the point of the lower boundary of the beam which is vertically under the load.

The integral expressions (71), (72), (73) transform as follows, if we write $y = y'' - b$,

$$\begin{aligned}
\mathrm{U} = &-\frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty\left\{\frac{2u\cosh 2u+\sinh 2u}{\sinh^2 2u-4u^2}\sin\frac{ux}{b}\cosh\frac{uy''}{b}-\frac{3}{4}\frac{x}{bu^2}\right\}du\\
&+\frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty\frac{2u\sinh 2u}{\sinh^2 2u-4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}\,du\\
&+\frac{\mathrm{W}}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty\left[\frac{2\sinh 2u}{\sinh^2 2u-4u^2}\sin\frac{ux}{b}\cosh\frac{uy''}{b}-\frac{3}{4}\frac{x}{bu^2}\right]du\\
&-\frac{\mathrm{W}}{2\pi}\frac{1}{\lambda'+\mu}\int_0^\infty\left\{\frac{2\cosh 2u+u^{-1}\sinh 2u}{\sinh^2 2u-4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}-\frac{3}{4}\frac{xy''}{b^2u^2}\right\}du\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{V} = &-\frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty\left\{\frac{2u\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b}-\frac{3}{4u^2}\right\}du\\
&+\frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty\left\{\frac{2u\cosh 2u+\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b}-\frac{3}{4}\frac{y''}{bu^2}\right\}du\\
&-\frac{\mathrm{W}}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty\left\{\frac{2\cosh 2u+u^{-1}\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b}\right.\\
&\qquad\qquad\left.-\frac{3}{4u^4}-\frac{3}{5u^2}-\frac{3}{8u^2}\frac{y''^2-x^2}{b^2}\right\}du\\
&+\frac{\mathrm{W}}{2\pi}\left(\frac{1}{\lambda'+\mu}\right)\int_0^\infty\left\{\frac{2\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b}-\frac{3}{4u^2}\frac{y''}{b}\right\}du\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}+\mathrm{Q} = &\frac{2\mathrm{W}}{\pi b}\int_0^\infty\left\{\frac{2u\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b}-\frac{3}{4u^2}\right\}du\\
&-\frac{2\mathrm{W}}{\pi b}\int_0^\infty\left\{\frac{2u\cosh 2u+\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b}-\frac{3}{4u^2}\frac{y''}{b}\right\}du\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}-\mathrm{Q} = &\frac{2\mathrm{W}}{\pi b}\int_0^\infty\left\{\frac{2u\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b}-\frac{3}{4u^2}\right\}du\\
&-\frac{2\mathrm{W}y''}{\pi b^2}\int_0^\infty\left\{\frac{2u^2\cosh 2u+u\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b}-\frac{3}{4u^2}\right\}du\\
&+\frac{2\mathrm{W}y''}{\pi b^2}\int_0^\infty\frac{2u^2\sinh 2u}{\sinh^2 2u-4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b}\,du\,,
\end{aligned}$$

$$S = \frac{W}{\pi b}\int_0^\infty \frac{2u\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}\,du$$
$$+\frac{Wy''}{\pi b^2}\int_0^\infty \frac{2u^2\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy''}{b}\,du$$
$$-\frac{Wy''}{\pi b^2}\int_0^\infty \frac{2u\cosh 2u + \sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}\,du.$$

These integrals remain convergent when we put $y'' = 0$, but they are not convergent in their present form for $y'' = 2b$.

If we expand now in powers of r'', where $x = r''\sin\phi''$, $y'' = r''\cos\phi''$, we obtain the following series, which can easily be shown to be uniformly and absolutely convergent inside a circle of radius $2b$:

$$\left.\begin{aligned}
U &= \frac{1}{\mu}\frac{2Wy''}{\pi b}\sum_1^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\sin\nu\phi''}{\nu!}H'_{\nu+1}\\
&\quad-\frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_1^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\sin\nu\phi''}{\nu!}H'_\nu+\frac{2W}{\pi}\frac{1}{\mu}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu+1}\frac{\sin\overline{2\nu+1}\phi''}{(2\nu+1)!}H'_{2\nu+1}\\
V &= -\frac{1}{\mu}\frac{2Wy''}{\pi b}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+1}\\
&\quad-\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_\nu-\frac{2W}{\pi}\frac{1}{\mu}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\cos 2\nu\phi''}{(2\nu)!}H'_{2\nu}
\end{aligned}\right\}\quad(87),$$

$$\left.\begin{aligned}
P &= \frac{4W}{\pi b}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+1}+\frac{4W}{\pi b}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\cos 2\nu\phi''}{(2\nu)!}H'_{2\nu+1}\\
&\quad-\frac{4Wy''}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+2}\\
Q &= -\frac{4W}{\pi b}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu+1}\frac{\sin\overline{2\nu+1}\phi''}{(2\nu+1)!}H'_{2\nu+2}+\frac{4Wy''}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+2}\\
S &= \frac{4W}{\pi b}\sum_1^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\sin 2\nu\phi''}{(2\nu)!}H'_{2\nu+1}-\frac{4Wy''}{\pi b^2}\sum_1^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\sin\nu\phi''}{\nu!}H'_{\nu+2}
\end{aligned}\right\}.\ (88),$$

where

H'_0 = an arbitrary constant depending upon the fixing conditions.

$$H'_1 = \int_0^\infty\left\{\frac{\frac{u}{2}\sinh 2u}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right\}du$$

$$H'_2 = \int_0^\infty\left\{\frac{\frac{1}{2}u^2\cosh 2u + \frac{1}{4}u\sinh 2u}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right\}du$$

$$H'_{2\nu+1} = \int_0^\infty \frac{\frac{1}{2}u^{2\nu+1} \sinh 2u}{\sinh^2 2u - 4u^2}\, du \ (\nu > 0)$$

$$H'_{2\nu} = \int_0^\infty \frac{\frac{1}{2}u^{2\nu} \cosh 2u + \frac{1}{4}u^{2\nu-1} \sinh 2u}{\sinh^2 2u - 4u^2}\, du \ (\nu > 1).$$

The values of the first few odd H′s are all we shall require. They are $H'_1 = -\cdot 049$, $H'_3 = +\cdot 537$, $H'_5 = +1\cdot 951$.

We then find, for points along the bottom edge, $y'' = 0$, $\phi'' = \pi/2$,

$$Q = 0, \ S = 0$$

$$P = \frac{8W}{\pi b}\left(-\cdot 049 - \frac{x^2}{b^2}\frac{1}{2\,!}(\cdot 537) + \frac{x^4}{b^4}\frac{1}{4\,!}(1\cdot 951) + \ldots\right).$$

This gives therefore a horizontal pressure at the point $(0, -b)$ equal to $\frac{W}{2b}(\cdot 250)$, and this pressure increases at a fairly rapid rate as we move away from the axis of y.

The stress P, obtained from BOUSSINESQ's calculation on STOKES' hypothesis, gives for the same point $P = \frac{W}{2b}\left(\frac{4}{\pi} - \frac{6}{\pi}\right) = -\frac{W}{2b}(\cdot 657)$. This value is considerably too high. We gather that STOKES' hypothesis ceases to give valid results for the points in the lower half of the beam.

§ 22. *Effect of Distributing the Concentrated Load over a small Area instead of a Line.*

In all the above work we have supposed the load W concentrated upon a line perpendicular to the plane of the strain. This has led us to expressions which make the stresses, and one displacement, infinite at the line where the load is applied, and the other displacement indeterminate. In practice, however, owing to the elasticity and plasticity of the materials both of the beam and of the knife-edge, contact along a geometrical line is impossible, and the load always distributes itself over an area, small but finite.

In the present section we shall therefore consider the effect of a uniform distribution of load W per unit area (W was formerly load per unit length), extending on either side of $x = 0$, $y = b$ for a distance a'.

Every line element $Wd\xi$ of this load at distance ξ from the middle will produce a system of stresses and displacements $Pd\xi$, $Qd\xi$, $Sd\xi$, $Ud\xi$, $Vd\xi$, such as we have just been investigating, except that for x we must write $(x - \xi)$.

The stresses and displacements due to the total load are therefore $\int_{-a'}^{+a'} P(x-\xi)\, d\xi$, $\int_{-a'}^{+a'} Q(x-\xi)\, d\xi$, $\int_{-a'}^{+a'} S(x-\xi)\, d\xi$, $\int_{-a'}^{+a'} U(x-\xi)\, d\xi$, $\int_{-a'}^{+a'} V(x-\xi)\, d\xi$, $P(x-\xi)$

denoting that $x - \xi$ is substituted for x in P. Similarly for Q, &c.; or writing $x - \xi = x'$, we have

$$P' = \int_{x-a'}^{x+a'} P(x')\,dx' \qquad U' = \int_{x-a'}^{x+a} U(x')\,dx'$$

$$Q' = \int_{x-a'}^{x+a'} Q(x')\,dx' \qquad V' = \int_{x-a'}^{x+a'} V(x')\,dx'$$

$$S' = \int_{x-a'}^{x+a'} S(x')\,dx'$$

P′, Q′, S′, U′, V′ referring to the stresses and displacements due to the uniform layer.

We can obtain in this way, at once, as many different forms for P′, Q′, S′, U′, V′ as we had for P, Q, S, U, V. The series for the latter integrate at once, for they are composed of terms of the form const. $\times\, r^n \cos n\phi$ or $r^n \sin n\phi$, or $yr^n \cos n\phi$ or $yr^n \sin n\phi$, where $r = \sqrt{x^2 + y^2}$, $\tan \phi = x/y$. We have then

$$\int r^n \sin n\phi\, dx = -\frac{1}{n+1} r^{n+1} \cos \overline{n+1}\,\phi$$

$$\int r^n \cos n\phi\, dx = \frac{1}{n+1} r^{n+1} \sin \overline{n+1}\,\phi.$$

The only case where this fails is when $n = -1$, and in this case it is easy to show that $\int \frac{\cos\phi}{r}\,dx = \phi$, $\int \frac{\sin\phi}{r}\,dx = \log r$.

Terms of the form ϕ and $\log r$ also occur. They can be integrated as follows:—

$$\int \phi\, dx = x\phi - y \log r,$$

$$\int \log r\, dx = x \log r - x + y\phi.$$

If we apply these formulæ, and if we call D_1 and D_2 (fig. ii.) the points $(-a', +b)$ and $(+a', +b)$, *i.e.*, the extremities of the layer of stress, and if r_1, r_2 denote the

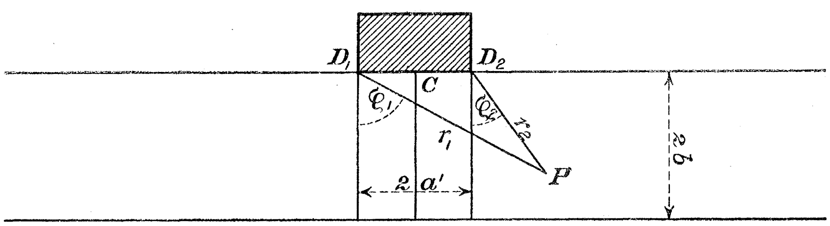

Fig. ii.

distances of any point from D_1, D_2 respectively, and if ϕ_1, ϕ_2 be the angles which r_1, r_2 make with the vertical, we find, if we start with the expressions for U, V, R, Q, S in the form (77), (79), (82), (84),

$$
\left.\begin{aligned}
U' = {} & \frac{1}{\mu}\frac{W}{2\pi}y'\log\frac{r_1}{r_2} - \frac{W}{2\pi}\frac{1}{\lambda'+\mu}\{(a'+x)\,\phi_1 - (x-a')\,\phi_2 - y'\log(r_1/r_2)\} \\
& + \frac{2Wy'}{\mu\pi}\sum_1^\infty(-1)^\nu H_\nu\frac{r_1^{\nu+1}\cos\overline{\nu+1}\,\phi_1 - r_2^{\nu+1}\cos\overline{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
& + \frac{2Wb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty H_{2\nu}\frac{r_1^{2\nu+2}\cos\overline{2\nu+2}\,\phi_1 - r_2^{2\nu+2}\cos\overline{2\nu+2}\,\phi_2}{b^{2\nu+2}(2\nu+2)!} \\
& - \frac{2Wb}{\pi}\frac{1}{\lambda'+\mu}\sum_1^\infty H_{2\nu-1}\frac{r_1^{2\nu+1}\cos\overline{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\cos\overline{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!} \\
V' = {} & -\frac{1}{\mu}\frac{W}{2\pi}y'(\phi_1-\phi_2) + \frac{W}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\Big\{(a'+x)\log\frac{r_1}{b\beta} \\
& \qquad - (x-a')\log\left(\frac{r_2}{b\beta}\right) + y'(\phi_1-\phi_2)\Big\} \\
& + 2B_1a' - \frac{2Wy'}{\mu\pi}\sum_0^\infty(-1)^\nu H_\nu\frac{r_1^{\nu+1}\sin\overline{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\overline{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
& - \frac{2Wb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left(2aH_{-1} + \sum_1^\infty H_{2\nu-1}\frac{r_1^{2\nu+1}\sin\overline{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\sin\overline{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!}\right) \\
& + \frac{2Wb}{\pi}\frac{1}{\lambda'+\mu}\sum_0^\infty\frac{r_1^{2\nu+2}\sin\overline{2\nu+2}\,\phi_1 - r_2^{2\nu+2}\sin\overline{2\nu+2}\,\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu}
\end{aligned}\right\} \quad (89).
$$

$$
\left.\begin{aligned}
P' = {} & -\frac{W}{\pi}(\phi_1-\phi_2) + \frac{W}{2\pi}(\sin 2\phi_1 - \sin 2\phi_2) \\
& - \frac{4W}{\pi}\sum_0^\infty H_{2\nu}\frac{r_1^{2\nu+1}\sin\overline{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\sin\overline{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!} \\
& - \frac{4W}{\pi}\sum_0^\infty(-1)^\nu H_\nu\frac{r_1^{\nu+1}\sin\overline{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\overline{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
& + \frac{4Wy'}{\pi b}\sum_0^\infty(-1)^\nu H_{\nu+1}\frac{(r_1^{\nu+1}\sin\overline{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\overline{\nu+1}\,\phi_2)}{b^{\nu+1}(\nu+1)!} \\
Q' = {} & -\frac{W}{\pi}(\phi_1-\phi_2) - \frac{W}{2\pi}(\sin 2\phi_1 - \sin 2\phi_2) \\
& + \frac{4W}{\pi}\sum_0^\infty H_{2\nu+1}\frac{r_1^{2\nu+2}\sin\overline{2\nu+2}\,\phi_1 - r_2^{2\nu+2}\sin\overline{2\nu+2}\,\phi_2}{b^{2\nu+2}(2\nu+2)!} \\
& - \frac{4Wy'}{\pi b}\sum_0^\infty(-1)^\nu H_{\nu+1}\frac{r_1^{\nu+1}\sin\overline{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\overline{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
S' = {} & -\frac{W}{2\pi}(\cos 2\phi_1 - \cos 2\phi_2) \\
& - \frac{4W}{\pi}\sum_1^\infty H_{2\nu}\frac{r_1^{2\nu+1}\cos\overline{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\cos\overline{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!} \\
& + \frac{4Wy'}{\pi b}\sum_1^\infty H_{\nu+1}(-1)^\nu\frac{r_1^{\nu+1}\cos\overline{\nu+1}\,\phi_1 - r_2^{\nu+1}\cos\overline{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!}
\end{aligned}\right\} \quad . \quad . \quad (90).
$$

The expressions (89) and (90) show us that in this case the stresses and the displacements are obtained as the difference of two functions taken with the extremities of the layer as origins. The series are everywhere uniformly and absolutely convergent inside the common part of two circles of radius $4b$ described about each of these extremities as centre. It follows that if these series are to be valid anywhere, the length of the layer must not exceed $8b$. And if they are to be valid round each extremity the length of the layer must be less than $4b$. If these conditions be not fulfilled, then we have to fall back on the results (74), (75), (76), and (83) for P, Q, S, U, V. Integrating these we obtain formulæ valid over the whole beam, and these again may be expanded in powers about any point we please, as has been previously shown. The results are rather long and do not seem to present sufficient interest to justify the writing out of them at length.

Assuming $2a' < 4b$, so that the expressions (89) and (90) are valid over a region enclosing the layer of application of the load, we see that here no displacement is either infinite or discontinuous. For in the limit, both $(a' + x) \log r_1$ and $y' \log r_1$ are zero when $x = - a'$, $y' = 0$; and in like manner $(x - a') \log r_2$ and $y' \log r_2$ are zero when $x = + a'$, $y' = 0$.

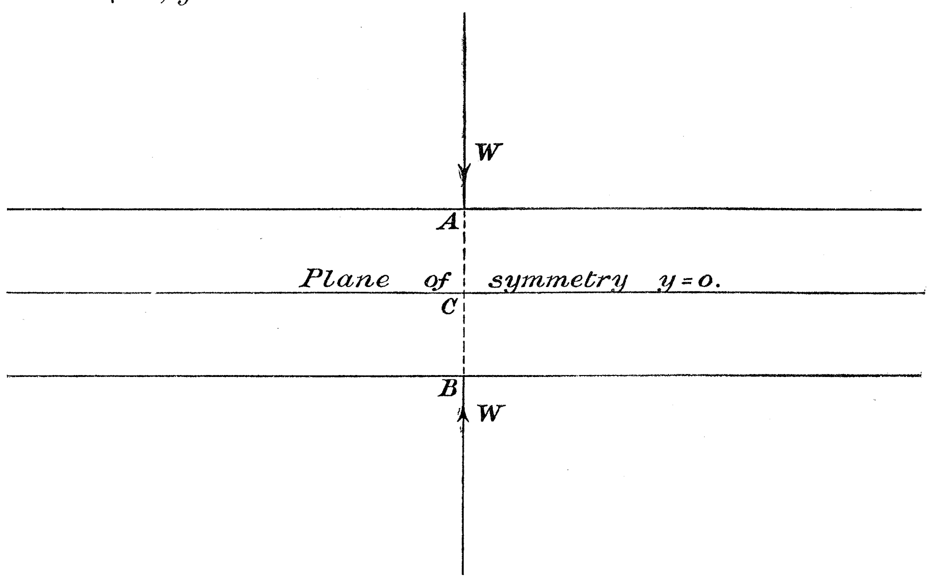

Fig. iii.

The shear S′ is continuous.

The stresses P′, Q′ however are discontinuous at the extremities of the layer. This indeed is obvious in the case of Q′, since it is one of the data of the problem. But it is curious to note that P′ is discontinuous at those points by precisely the same amount as Q′.

§ 23. *Case of a Beam under Two Equal and Opposite Loads, or Resting upon a Rigid Smooth Plane.*

If we take the solution we have obtained, turn it upside down, as it were, and superpose it to itself, we obtain the solution of the problem of an infinitely long beam gripped between two knife-edges exactly opposite each other (fig. iii.). The solution is obtained from the previous one by changing the signs of y, V and S, and then adding the new U, V, P, Q, S to the old.

I do not propose to write down fully the solution; it is easily obtained in various forms by using the several expansions which have already been given for the beam under a single concentrated load only. The parts of the stresses and displacements which become infinite at the points of loading are of exactly the same form as in the previous case.

Let us, however, consider the stresses. We easily find the following expressions:

$$\left.\begin{aligned} P &= -\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ &\quad -\frac{2W}{\pi b}\int_0^\infty \frac{uy}{b}\,\frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du. \\ Q &= -\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ &\quad +\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du. \\ S &= \frac{2W}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ &\quad -\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du. \end{aligned}\right\} \quad \ldots\ldots \quad (91).$$

The last written equation shows that $S = 0$ over the plane $y = 0$. Further, from considerations of symmetry $V = 0$ over this plane. Hence we may, if we choose, leave the lower part of the beam out of account altogether, and consider it as replaced by an infinite smooth rigid plane, against which the beam is pressed by a single weight, W. It then becomes of considerable interest to find out how this weight W distributes itself, after transmission through the beam, over this rigid plane.

The pressure $-$ Q on the plane corresponding to $y = 0$ is given by

$$-Q = +\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\,du. \quad \ldots\ldots \quad (92).$$

It is easy to show that this pressure tends to zero when x is large.

Integrating by parts with regard to u, we have

$$Q = \frac{2W}{\pi x}\int_0^\infty \frac{d}{du}\left(\frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\right)\sin\frac{ux}{b}\,du.$$

The integral on the right-hand side is obviously not infinite, however large x may be. Hence Q tends to zero as x tends to infinity.

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We might repeat this process any finite number of times. It will be found that $\frac{\sinh u + u\cosh u}{\sinh 2u + 2u}$ being an even function of u, the integrated terms will in all cases vanish at both limits, and we obtain $Q = \frac{2Wb^n}{\pi x^{n+1}} \times$ an integral which is not infinite when x is large. Therefore we see that Q diminishes faster than any finite inverse power of x, however high. This seems to suggest an exponential law.

§ 24. *New Form of Expansion for the Pressure on the Rigid Plane.*

Consider the integral

$$I = \int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u} \cos uz \, du.$$

We have

$$\frac{1}{\sinh 2u + 2u} = \frac{1}{\sinh 2u} - \frac{2u}{\sinh^2 2u} + \ldots + (-1)^r \frac{(2u)^r}{\sinh^{r+1} 2u} + \ldots$$

$$+ (-1)^{n-1} \frac{(2u)^{n-1}}{\sinh^n 2u} + (-1)^n \frac{(2u)^n}{\sinh^n 2u\,(\sinh 2u + 2u)}.$$

Substitute in I, we find

$$I = J_0 + J_1 + \ldots + J_r + \ldots + J_{n-1} + R_n,$$

where

$$J_r = (-1)^r \int_0^\infty \frac{(2u)^r}{\sinh^{r+1} 2u} (\sinh u + u\cosh u) \cos uz \, du,$$

$$R_n = (-1)^n \int_0^\infty \frac{(2u)^n}{\sinh^n 2u} \frac{(\sinh u + u\cosh u)}{\sinh 2u + 2u} \cos uz \, du.$$

Now

$$J_r = (-1)^r 2^{2r+1} \int_0^\infty \frac{u^r (\sinh u + u\cosh u)}{e^{2(r+1)u}(1 - e^{-4u})^{r+1}} \cos uz \, du.$$

Let us assume that in this we may expand $(1 - e^{-4u})^{r+1}$ in ascending powers of e^{-4u}. This will be justified later.

$$\frac{1}{(1 - e^{-4u})^{r+1}} = \sum_{s=0}^{s=\infty} \frac{(s+1)\ldots(s+r)}{r!} e^{-4su},$$

whence

$$J_r = (-1)^r 2^{2r} \int_0^\infty u^r \sum_{s=0}^{s=\infty} \frac{(s+1)\ldots(s+r)}{r!} \{e^{-(4s+2r+1)u} - e^{-(4s+2r+3)u}\} \cos uz \, du$$

$$+ (-1)^r 2^{2r} \int_0^\infty u^{r+1} \sum_{s=0}^{s=\infty} \frac{(s+1)\ldots(s+r)}{r!} \{e^{-(4s+2r+1)u} + e^{-(4s+2r+3)u}\} \cos uz \, du.$$

The cases r even and r odd have to be treated separately. Consider first r even and $= 2t$, and let K_r and L_r denote the first and second integrals in the last written expression for J_r. Then owing to the vanishing factor we may take the Σ in K_r as going back to $s = -t$, or, putting $s' = s + t$,

$$K_{2t} = 2^{4t}\int_0^\infty u^{2t}\sum_{s'=0}^{s'=\infty}\frac{(s'-t+1)\ldots(s'-1)s'\ldots(s'+t)}{(2t)!}\{e^{-(4s'+1)u}-e^{-(4s'+3)u}\}\cos uz\,du.$$

But

$$\begin{aligned}
4s'-4t+4 &= (4s'+1)-(4t-3)\\
&\ldots\ldots\\
4s'-4 &= (4s'+1)-5\\
4s' &= (4s'+1)-1\\
4s'+4 &= (4s'+1)+3\\
&\ldots\ldots\\
4s'+4t &= (4s'+1)+(4t-1),
\end{aligned}$$

and similarly

$$\begin{aligned}
4s'-4t+4 &= (4s'+3)-(4t-1)\\
&\ldots\ldots\\
4s'-4 &= (4s'+3)-7\\
4s' &= (4s'+3)-3\\
4s'+4 &= (4s'+3)+1\\
&\ldots\ldots\\
4s'+4t &= (4s'+3)+(4t-3).
\end{aligned}$$

Now let $a_0, a_1, \ldots a_{2t}$ be the coefficients in the product of degree $2t$

$$\{x+(4t-1)\}\{x+(4t-5)\}\ldots\{x-(4t-3)\},$$

when it is expanded out, so that this product is

$$a_0x^{2t}+a_1x^{2t-1}+\ldots+a_{2t}.$$

Then

$$\{x-(4t-1)\}\{x-(4t-5)\}\ldots\{x+(4t-3)\}$$

$$= a_0x^{2t}-a_1x^{2t-1}+\ldots+a_{2t}.$$

K_{2t} may then be written

$$\int_0^\infty \frac{u^{2t}}{(2t)!}\left[\begin{aligned}
&\sum_{s'=0}^{s'=\infty} a_0\{(4s'+1)^{2t}e^{-\widehat{4s'+1}u}-(4s'+3)^{2t}e^{-\widehat{4s'+3}u}\}\\
+&\sum_{s=0}^{s'=\infty} a_1\{(4s'+1)^{2t-1}e^{-\widehat{4s'+1}u}+(4s'+3)^{2t-1}e^{-\widehat{4s'+3}u}\}\\
+&\sum_{s'=0}^{s'=\infty} a_2\{(4s'+1)^{2t-2}e^{-\widehat{4s'+1}u}-(4s'+3)^{2t-2}e^{-\widehat{4s'+3}u}\}\\
+&\ldots\ldots\ldots\\
+&\sum_{s'=0}^{s'=\infty} a_{2t-1}\{(4s'+1)e^{-\widehat{4s'+1}u}+(4s'+3)e^{-\widehat{4s'+3}u}\}\\
+&\sum_{s'=0}^{s'=\infty} a_{2t}\{e^{-\widehat{4s'+1}u}-e^{-\widehat{4s'+3}u}\}
\end{aligned}\right]\cos uz\,du.$$

Now if, as we have assumed, our expansion of $(1 - e^{-4u})^{-r-1}$ was justifiable, we may stop at the ν^{th} term, leaving a remainder less than an assigned quantity, provided ν be taken large enough. It will be shown in the next article that this is the case.

We may then, in the above, write for the upper limit of s' a number ν, large but finite. The series now consisting of a finite number of terms, we may distribute the integral sign, and further, we can replace u^{2t} by $(-1)^t \dfrac{d^{2t}}{dz^{2t}}$, since obviously each of the integrals of the type $\int_0^\infty e^{-ku} \cos uz \, du$ when $k > 0$ allows of being differentiated under the integral sign. This gives us, when the several integrals are evaluated,

$$K_{2t} = \frac{(-1)^t}{(2t)!} \frac{d^{2t}}{dz^{2t}} \left[\begin{aligned} & a_0 \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)^{2t+1}}{(4s'+1)^2 + z^2} - \frac{(4s'+3)^{2t+1}}{(4s'+3)^2 + z^2} \right\} \\ + & a_1 \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)^{2t}}{(4s'+1)^2 + z^2} + \frac{(4s'+3)^{2t}}{(4s'+3)^2 + z^2} \right\} \\ + & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ + & a_{2t-1} \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)^2}{(4s'+1)^2 + z^2} + \frac{(4s'+3)^2}{(4s+3)^2 + z^2} \right\} \\ + & a_{2t} \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)}{(4s'+1)^2 + z^2} - \frac{(4s'+3)}{(4s'+3)^2 + z^2} \right\} \end{aligned} \right].$$

Now writing in the above

$$(4s'+1)^2 = \{(4s'+1)^2 + z^2\} - z^2, \; (4s'+3)^2 = \{(4s'+3)^2 + z^2\} - z^2,$$

and remembering that d^{2t}/dz^{2t} destroys any power of $z < 2t$, we find

$$K_{2t} = \frac{(-1)^t}{(2t)!} \frac{d^{2t}}{dz^{2t}} \left[\begin{aligned} & (a_0 (-1)^t z^{2t} + a_2 (-1)^{t-1} z^{2t-2} + \ldots \\ & \qquad + a_{2t}) \sum_{s'=0}^{s'=\nu} \left\{ \frac{4s'+1}{(4s'+1)^2 + z^2} - \frac{4s'+3}{(4s'+3)^2 + z^2} \right\} \\ + & (a_1 (-1)^t z^{2t} + a_3 (-1)^{t-1} z^{2t-2} + \ldots \\ & \qquad - a_{2t-1} z^2) \sum_{s'=0}^{s'=\nu} \left\{ \frac{1}{(4s'+1)^2 + z^2} + \frac{1}{(4s'+3)^2 + z^2} \right\} \end{aligned} \right].$$

But, from CHRYSTAL'S 'Algebra,' vol. 2, p. 338,

$$\frac{\pi}{4z} \tanh \frac{\pi z}{2} = \sum_{s'=0}^{s'=\infty} \left\{ \frac{1}{(4s'+1)^2 + z^2} + \frac{1}{(4s'+3)^2 + z^2} \right\}$$

$$\frac{\pi}{4} \operatorname{sech} \frac{\pi z}{2} = \sum_{s'=0}^{s'=\infty} \left\{ \frac{4s'+1}{(4s'+1)^2 + z^2} - \frac{4s'+3}{(4s'+3)^2 + z^2} \right\}.$$

If, therefore, in our expression for K_{2t} we now allow ν to increase indefinitely, we obtain

$$K_{2t} = \frac{\pi}{4} \frac{(-)^t}{(2t)!} \left\{ \frac{d^{2t}}{dz^{2t}} \left[\psi_{2t}(z) \operatorname{sech} \frac{\pi z}{2} + \chi_{2t}(z) \tanh \frac{\pi z}{2} \right] \right\},$$

where

$$\psi_{2t}(z) = a_0(-1)^t z^{2t} + a_2(-1)^{t-1} z^{2t-2} + \ldots + a_{2t}$$

$$= \tfrac{1}{2}\left[\begin{array}{l}(\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5)\ldots(\sqrt{-1}z - \widehat{4t-3}) \\ +(-\sqrt{-1}z + 4t - 1)(-\sqrt{-1}z + 4t - 5)\ldots(-\sqrt{-1}z - \widehat{4t-3})\end{array}\right]$$

$$\chi_{2t}(z) = a_1(-1)^t z^{2t-1} + \ldots - a_{2t-1}z$$

$$= \frac{\sqrt{-1}}{2}\left[\begin{array}{l}(\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5)\ldots(\sqrt{-1}z - \widehat{4t-3}) \\ -(-\sqrt{-1}z + 4t - 1)(-\sqrt{-1}z + 4t - 5)\ldots(-\sqrt{-1}z - \widehat{4t+3})\end{array}\right].$$

If we treat in a precisely similar way the second integral L_{2t}, we find

$$L_{2t} = \frac{\pi}{4}\frac{(-)^t}{(2t)!}\frac{d^{2t+1}}{dz^{2t+1}}\left(\psi_{2t}(z)\tanh\frac{\pi z}{2} - \chi_{2t}(z)\,\text{sech}\,\frac{\pi z}{2}\right).$$

Coming now to the case where $r = \text{odd} = 2t + 1$, we work out K_{2t+1} and L_{2t+1} by a similar method. We consider in this case the product of degree $2t + 1$,

$$(x + 4t - 1)(x + 4t - 5)\ldots(x - \widehat{4t-3})(x - \widehat{4t+1}),$$

which we denote by

$$b_0 x^{2t+1} + b_1 x^{2t} + b_2 x^{2t-1} + \ldots + b_{2t+1}.$$

After reductions of the same type as those used for K_{2t}, we find

$$K_{2t+1} = \frac{(-1)^t}{(2t+1)!}\frac{d^{2t+1}}{dz^{2t+1}}\left[\begin{array}{l}\quad(b_0(-1)^t z^{2t+1} + b_2(-1)^{t-1}z^{2t-1} + \ldots \\ \qquad\qquad + b_{2t}z)\displaystyle\sum_{s'=0}^{s'=\nu}\left\{\frac{4s'+1}{(4s'+1)^2+z^2} - \frac{4s'+3}{(4s'+3)^2+z^2}\right\} \\ +(b_1(-1)^t z^{2t+1} + b_3(-1)^{t-1}z^{2t-1} + \ldots \\ \qquad\qquad + b_{2t+1}z)\displaystyle\sum_{s'=0}^{s'=\nu}\left\{\frac{1}{(4s'+1)^2+z^2} + \frac{1}{(4s'+3)^2+z^2}\right\}\end{array}\right].$$

$$L_{2t+1} = \frac{(-1)^t}{(2t+1)!}\frac{d^{2t+2}}{dz^{2t+2}}\left[\begin{array}{l}\quad(b_0(-1)^{t+1}z^{2t+2} + \ldots \\ \qquad\qquad - b_{2t}z^2)\displaystyle\sum_{s'=0}^{s'=\nu}\left\{\frac{1}{(4s'+1)^2+z^2} + \frac{1}{(4s'+3)^2+z^2}\right\} \\ +(b_1(-1)^t z^{2t} + \ldots \\ \qquad\qquad + b_{2t+1})\displaystyle\sum_{s'=0}^{s'=\nu}\left\{\frac{4s'+1}{(4s'+1)^2+z^2} - \frac{4s'+3}{(4s'+3)^2+z^2}\right\}\end{array}\right],$$

whence writing

$$b_0(-1)^t z^{2t+1} + b_2(-1)^{t-1} z^{2t-1} + \ldots + z b_{2t}$$

$$= \frac{1}{2\sqrt{-1}} \left[\begin{array}{l} (\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5)\ldots \\ \qquad (\sqrt{-1}z - \widehat{4t-3})(\sqrt{-1}z - \widehat{4t+1}) \\ + (\sqrt{-1}z - \widehat{4t-1})(\sqrt{-1}z - \widehat{4t-5})\ldots \\ \qquad (\sqrt{-1}z + \widehat{4t-3})(\sqrt{-1}z + \widehat{4t+1}) \end{array} \right] = \psi_{2t+1}(z)$$

$$b_1(-1)^t z^{2t} + b_3(-1)^{t-1} z^{2t-2} + \ldots + b_{2t+1}$$

$$= \tfrac{1}{2} \left[\begin{array}{l} (\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5)\ldots \\ \qquad (\sqrt{-1}z - \widehat{4t-3})(\sqrt{-1}z - \widehat{4t+1}) \\ - (\sqrt{-1}z - \widehat{4t-1})(\sqrt{-1}z - \widehat{4t-5})\ldots \\ \qquad (\sqrt{-1}z + 4t-3)(\sqrt{-1}z + 4t+1) \end{array} \right] = \chi_{2t+1}(z),$$

we obtain

$$K_{2t+1} = \frac{(-1)^t}{(2t+1)!}\,\frac{\pi}{4}\,\frac{d^{2t+1}}{dz^{2t+1}}\left[\psi_{2t+1}(z)\operatorname{sech}\frac{\pi z}{2} + \chi_{2t+1}(z)\tanh\frac{\pi z}{2}\right]$$

$$L_{2t+1} = \frac{(-1)^t}{(2t+1)!}\,\frac{\pi}{4}\,\frac{d^{2t+2}}{dz^{2t+2}}\left[-\psi_{2t+1}(z)\tanh\frac{\pi z}{2} + \chi_{2t+1}(z)\operatorname{sech}\frac{\pi z}{2}\right],$$

and since $J_r = K_r + L_r$, we find that the required integral

$$I = \frac{\pi}{4}\sum_{t=0}^{t=n}\frac{(-1)^t}{(2t)!}\,\frac{d^{2t}}{dz^{2t}}\left\{\begin{array}{l}\psi_{2t}(z)\operatorname{sech}\dfrac{\pi z}{2} + \chi_{2t}(z)\tanh\dfrac{\pi z}{2} \\ + \dfrac{d}{dz}\left(\psi_{2t}(z)\tanh\dfrac{\pi z}{2} - \chi_{2t}(z)\operatorname{sech}\dfrac{\pi z}{2}\right)\end{array}\right\}$$

$$+ \frac{\pi}{4}\sum_{t=0}^{t=n}\frac{(-1)^t}{(2t+1)!}\,\frac{d^{2t+1}}{dz^{2t+1}}\left\{\begin{array}{l}\psi_{2t+1}(z)\operatorname{sech}\dfrac{\pi z}{2} + \chi_{2t+1}(z)\tanh\dfrac{\pi z}{2} \\ - \dfrac{d}{dz}\left(\psi_{2t+1}(z)\tanh\dfrac{\pi z}{2} - \chi_{2t+1}(z)\operatorname{sech}\dfrac{\pi z}{2}\right)\end{array}\right\}$$

$$+ R_{2n+2} \quad \ldots \ldots \quad (93).$$

§ 25. *Justification of the Procedure employed in the last Section.*

We have to show that, in the case of the integral

$$J_n = \int_0^\infty \frac{(2u)^n}{\sinh^{n+1} 2u}(\sinh u + u\cosh u)\cos uz\,du$$

$$= 2\int_0^\infty \frac{(4u)^n e^{-\widehat{2n+2}u}}{(1-e^{-4u})^{n+1}}(\sinh u + u\cosh u)\cos uz\,du$$

we were justified in expanding $(1-e^{-4u})^{-n-1}$ in ascending powers of e^{-4u}.

Now

$$\frac{1}{1-x} = 1 + x + \ldots + x^{n+r-1} + \frac{x^{n+r}}{1-x}.$$

Differentiate n times with regard to x,

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \ldots + \frac{r(r+1)\ldots(r+n-1)}{n!}x^{r-1} + \frac{1}{n!}\frac{d^n}{dx^n}\left(\frac{x^{n+r}}{1-x}\right).$$

The remainder is therefore

$$\frac{1}{n!}\frac{d^n}{dx^n}\frac{x^{n+r}}{(1-x)} = \frac{x^{n+r}}{(1-x)^{n+1}} + \ldots$$
$$+ \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}\frac{x^{n+r-s}}{(1-x)^{n+1-s}} + \ldots + \frac{(n+r)\ldots(r+1)}{n!}\frac{x^r}{1-x}.$$

This holds for all values of x however near to 1. Putting $x = e^{-4u}$ and substituting in J_n, we find J_n = 1st r terms of the series + a remainder term consisting of the sum of $(n+1)$ integrals of the form

$$2\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!} e^{-(6n+4r-4s+2)u}\frac{(4u)^n(\sinh u + u\cosh u)}{(1-e^{-4u})^{n+1-s}}\cos uz\,du,$$

s ranging from 0 to n, and the product $\dfrac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}$ being replaced by unity for $s = 0$.

Now $\sinh u$ is always $< u\cosh u$: hence the general integral in the remainder (the factor multiplying $\cos uz$ in the integrand being positive throughout) is less than

$$\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}(4u)^s e^{-(6n+4r-4s+2)u}\cosh u\left(\frac{4u}{1-e^{-4u}}\right)^{n+1-s}du,$$

i.e., than

$$\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}e^{-(4n+4r-2s)u}(4u)^s\cosh u\left(\frac{2u}{\sinh 2u}\right)^{n+1-s}du.$$

Now $\dfrac{2u}{\sinh 2u} < 1$ always, and $\cosh u < e^u$. The general remainder term is therefore less than

$$\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}(4u)^s e^{-(4n+4r-2s-1)u}du$$
$$< \tfrac{1}{4}\frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{\left(n+r-\frac{s}{2}-\frac{1}{4}\right)^{s+1}} \text{ for } s \text{ ranging from 1 to } n.$$

For $s = 0$ the remainder term $< \tfrac{1}{4}\dfrac{1}{\left(n+r-\frac{s}{2}-\frac{1}{4}\right)}$. Thus for every value of s the value of the corresponding term in the remainder is seen to become very small of the order $1/r$ when r is made very large, n remaining finite. The value of the whole remainder is therefore also small of the order $1/r$. Consequently this remainder tends to zero as we make r large, and the series is therefore a true arithmetical equivalent of J_n.

We have still to show that a similar result holds for the expansion found for I, namely, that the integral we have called R_n tends to the limit zero when n is indefinitely increased. This we can do as follows:—

$$R_n = (-1)^n \int_0^\infty \frac{(2u)^n}{\sinh^n 2u} \frac{\sinh u + u \cosh u}{\sinh 2u + 2u} \cos uz \, du$$

is numerically less than

$$\int_0^\infty \frac{(2u)^n}{\sinh^n 2u} \frac{\sinh u + u \cosh u}{\sinh 2u + 2u} \, du,$$

and it is easy to show that both $(2u)^n/\sinh^n 2u$ and $(\sinh u + u \cosh u)/(\sinh 2u + 2u)$ continually decrease as u increases.

Hence, if we split up $\int_0^\infty$ into $\int_0^\omega + \int_\omega^\infty$, the first part is less than $\{\omega \times$ value of the integrand when $u = 0\}$, *i.e.*, $< \omega/2$. The second part is also less than

$$\frac{(2\omega)^n}{(\sinh^n 2\omega)} \int_\omega^\infty \frac{\sinh u + u \cosh u}{\sinh 2u + 2u} \, du.$$

Denoting the last integral, which is finite, by M, we have $R_n < \frac{\omega}{2} + \frac{(2\omega)^n}{(\sinh^n 2\omega)} M$ numerically.

But

$$\frac{(2\omega)^n}{\sinh^n 2\omega} < \frac{(2\omega)^n}{\left(2\omega + \frac{8\omega^3}{6}\right)^n} < \frac{1}{\left(1 + \frac{2\omega^2}{3}\right)^n} < \frac{1}{1 + \frac{2n\omega^2}{3}}.$$

Therefore $R_n < \frac{\omega}{2} + \frac{M}{1 + \frac{2n\omega^2}{3}}$.

Now if ω be chosen equal to $n^{-\frac{1}{3}}$, $R_n < \frac{1}{2n^{\frac{1}{3}}} + \frac{M}{1 + \frac{2}{3}n^{\frac{1}{3}}}$, a quantity which tends to zero when n tends to infinity. R_n itself therefore tends to zero for all values of z, so that the series (93) may be extended to infinity.

§ 26. *Deductions as to the Rapidity with which the Local Disturbances die out as we leave the neighbourhood of the Load.*

If we look at (93) and perform the differentiations, then, remembering that $\chi_{2t}(z)$ is of degree $(2t - 1)$ in z, $\chi_{2t+1}(z)$ and $\psi_{2t}(z)$ are of degree $2t$ in z, and $\psi_{2t+1}(z)$ is of degree $(2t + 1)$ in z, the only terms occurring in I will be of the form (algebraic polynomial in z) $\times \left(\text{sech} \frac{\pi z}{2} \text{ or } \text{sech}^2 \frac{\pi z}{2}, \text{ or their differential coefficients}\right)$. Now

$\operatorname{sech}\frac{\pi z}{2}$ and all its differential coefficients will be of order $e^{-\frac{\pi z}{2}}$ when z is large. Similarly $\operatorname{sech}^2\frac{\pi z}{2}$ and its differential coefficients will be of order $e^{-\pi z}$ when z is large. We see, therefore, that the first n terms of the series for I will be of the form (algebraic polynomial of degree n in z) $e^{-\frac{\pi z}{2}}$ to the first approximation when z is large.

Further we have obtained an expression for the remainder R_n, which is small independently of z, for any given large value of n. We see therefore that, n being assigned, we may make z as large as we please and I will eventually tend to zero, $e^{\pi z/2}$ becoming large more rapidly than any polynomial of finite degree, if z be large enough.

Now $z = x/b$. We see therefore that, if b be small, the pressure, after a certain value of x, decreases with extreme rapidity as we get away from the neighbourhood

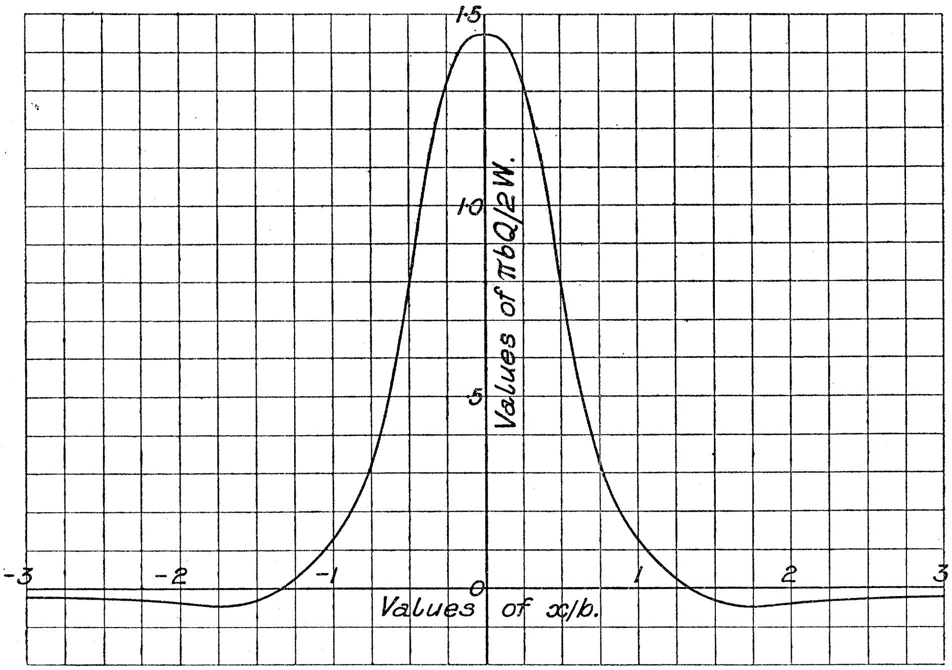

Fig. iv.

of the concentrated load, because, z being then large, even for moderate values of x the influence of the exponential term will be predominant. On the other hand, if b becomes finite, or even large, the algebraic polynomial factor will become predominant, and the decrease as we go away from the point of loading will become much less rapid. The expansion (93) gives us a link, as it were, between the case of a very thin beam, where the local effects die out according to a negative exponential of the distance along the axis, and that of an infinite solid, where they decrease as an inverse power of the distance from the point of loading.

A diagram is given in fig. iv. showing the variation of the pressure Q along the base of the elastic block where it rests on the rigid plane. The ordinates represent the ratio of Q to $2W/\pi b$—that is, the integral which has been called I. The abscissæ represent the quantities x/b. The diagram has been plotted from the following values of I, which have been calculated :—

x/b.	I.
0	1·4444
$\pi/6$	·7412
$\pi/3$	·1125
$\pi/2$	– ·0300
$2\pi/3$	– ·0252
π	– ·0036

For a value of x/b equal to 1·35 about the pressure vanishes, and is replaced by a *tension*. This is a very remarkable result, as it shows that an elastic block, acted upon by a concentrated load along a line of its upper surface transverse to its length, cannot have its whole base in contact with a smooth rigid plane on which it rests: at a certain distance from the load the body of the beam is lifted off the plane.

It would therefore appear as though the problem treated of above were impossible to realise in practice. But obviously we may superimpose any uniform pressure on the top of the beam, sufficient to make the total pressure at every point below positive. This may be done, in some cases, by the weight of the beam itself, if the weight W be not too large.

Further, the tensions required to keep the lower surface of the block horizontal are, as we may see from fig. iv., very small. If we leave them out of account, we do not sensibly disturb the distribution of the large pressure under the load, so that fig. iv. still gives us an approximation if we omit the negative part of the curve altogether.

This gives a maximum pressure just below the load equal to $(W/b) \times$ ·920, or rather less than the pressure due to the load W distributed uniformly over the vertical cross-section of the block. This pressure diminishes rapidly as we go away from this point, being very small at a distance from it equal to about 1·35 of the height of the block.

We cannot tell exactly, in the actual case, where the pressure will be first absolutely nil. We can form a rough estimate, however, of the dimensions of the area in contact by taking the area over which, in the solution obtained, the stress is always a pressure. This area extends to a distance of 1·35 $\times$ height of block, on either side of the vertical through the load.

Some rough experiments on a block of india-rubber lying on a wooden table have confirmed the result that the block is lifted out of contact with the table away from the load, and that the area of contact is of the above order.

PART III.

SOLUTION FOR A BEAM UNDER ASYMMETRICAL NORMAL FORCES: SPECIAL CASE OF TWO OPPOSITE CONCENTRATED LOADS NOT IN THE SAME VERTICAL STRAIGHT LINE.

§ 27. *Expressions for the Displacements and Stresses in Series.*

Let us now proceed to consider what the general solution becomes in the case of a beam subject to normal forces which are now no longer restricted to be symmetrical.

In this case coefficients γ and δ come in, as well as α, β; κ, ν, ζ, θ being all zero.

Consider particularly a beam (fig. v.) subject to a downwards concentrated

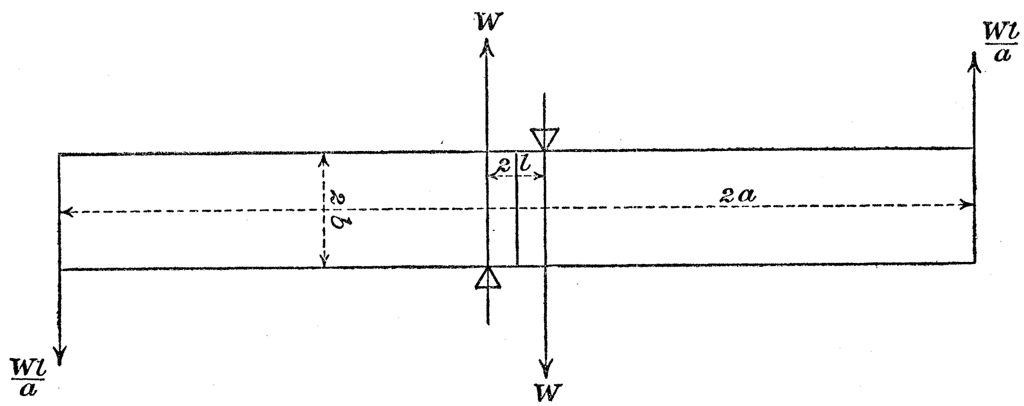

Fig. v.

load W, acting upon its upper surface at $x = l$, and an upwards concentrated load W, acting upon its lower surface at $x = -l$.

Such a system by itself is not in equilibrium. But the solution will introduce two shears over the ends, equal to $\sum_1^\infty (\gamma_n - \delta_n) \frac{\cos ma}{m}$ by equation (50).

In the case taken above $\alpha_0 = \beta_0 = -W/2a$, $\alpha_n = \beta_n = -\frac{W}{a}\cos ml, \gamma_n = -\delta_n$ $= -\frac{W}{a}\sin ml$, where $m = n\pi/a$, n being an integer. Hence the shears over the ends are $\sum_1^\infty (-1)^{n-1}\frac{2W}{n\pi}\sin\frac{n\pi l}{a} = \frac{Wl}{a}$, and these will satisfy the conditions of rigid equilibrium.

We then find the following expressions for the stresses and displacements in series:

$$
\left.\begin{aligned}
U = &- \sum_1^\infty \frac{1}{m}\frac{W}{a}\cos ml \frac{\left\{\frac{1}{\lambda'+\mu}\sinh mb - \frac{1}{\mu} mb\cosh mb\right\}}{\sinh 2mb + 2mb}\cosh my \sin mx \\
&- \sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\cos ml \sinh mb}{\sinh 2mb + 2mb}\, y \sinh my \sin mx \\
&+ \sum_1^\infty \frac{1}{m}\frac{W}{a}\sin ml \frac{\left\{\frac{1}{\lambda'+\mu}\cosh mb - \frac{1}{\mu} mb\sinh mb\right\}}{\sinh 2mb - 2mb}\sinh my \cos mx \\
&+ \sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\sin ml \cosh mb}{\sinh 2mb - 2mb}\, y \cosh my \cos mx \\
&- \frac{\eta W x}{2aE} - Ay. \\
V = &- \sum_1^\infty \frac{1}{m}\frac{W}{a}\cos ml \frac{\left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sinh mb + \frac{1}{\mu} mb\cosh mb\right\}}{\sinh 2mb + 2mb}\sinh my \cos mx \\
&+ \sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\cos ml \sinh mb}{\sinh 2mb + 2mb}\, y \cosh my \cos mx \\
&- \sum_1^\infty \frac{1}{m}\frac{W}{a}\sin ml \frac{\left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh mb + \frac{1}{\mu} mb\sinh mb\right\}}{\sinh 2mb - 2mb}\cosh my \sin mx \\
&+ \sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\sin ml \cosh mb}{\sinh 2mb - 2mb}\, y \sinh my \sin mx \\
&- \frac{Wy}{2aE} + Ax.
\end{aligned}\right\} \quad (94).
$$

$$
\left.\begin{aligned}
P = &- \frac{2W}{a}\sum_1^\infty \cos ml \frac{\sinh mb - mb\cosh mb}{\sinh 2mb + 2mb}\cos mx \cosh my \\
&- \frac{2W}{a}\sum_1^\infty \cos ml \frac{my \sinh mb}{\sinh 2mb + 2mb}\cos mx \sinh my \\
&- \frac{2W}{a}\sum_1^\infty \sin ml \frac{\cosh mb - mb\sinh mb}{\sinh 2mb - 2mb}\sin mx \sinh my \\
&- \frac{2W}{a}\sum_1^\infty \sin ml \frac{my \cosh mb}{\sinh 2mb - 2mb}\sin mx \cosh my. \\
Q = &- \frac{W}{2a} - \frac{2W}{a}\sum_1^\infty \cos ml \frac{\sinh mb + mb\cosh mb}{\sinh 2mb + 2mb}\cos mx \cosh my \\
&+ \frac{2W}{a}\sum_1^\infty \cos ml \frac{my \sinh mb}{\sinh 2mb + 2mb}\cos mx \sinh my \\
&- \frac{2W}{a}\sum_1^\infty \sin ml \frac{\cosh mb + mb\sinh mb}{\sinh 2mb - 2mb}\sin mx \sinh my \\
&+ \frac{2W}{a}\sum_1^\infty \sin ml \frac{my \cosh mb}{\sinh 2mb - 2mb}\sin mx \cosh my.
\end{aligned}\right\} \quad (95).
$$

$$\left.\begin{aligned} S = {} & \frac{2W}{a}\sum_1^\infty \cos ml \frac{mb\cosh mb}{\sinh 2mb + 2mb}\sin mx \sinh my \\ & - \frac{2W}{a}\sum_1^\infty \cos ml \frac{my\sinh mb}{\sinh 2mb + 2mb}\sin mx\cosh my \\ & - \frac{2W}{a}\sum_1^\infty \sin ml \frac{mb\sinh mb}{\sinh 2mb - 2mb}\cos mx\cosh my \\ & + \frac{2W}{a}\sum_1^\infty \sin ml \frac{my\cosh mb}{\sinh 2mb - 2mb}\cos mx\sinh my \end{aligned}\right\} \quad . \quad . \quad . \quad (96),$$

where A in the above is an arbitrary constant representing a rigid body rotation. If the conditions of fixing are that the two extremities of the horizontal axis are to remain at the same vertical height after strain, A is zero.

If, on the other hand, we fix the beam in such a way that the shears Wl/a over the ends are each allowed to produce, at the extremities of the axis, the deflection which they would produce if the bar were clamped at its middle and the deflection were calculated on the Euler-Bernoulli theory, then we find $Aa = \frac{Wla^2}{2Eb^3}$. This appears to be the more natural method of fixing. We shall, therefore, in what follows, suppose A to have this value.

§ 28. *Integral Expressions when a is made Infinite.*

When we increase the length of the bar indefinitely, it is easy to show that, if we take the last given value of A, the displacements remain finite at a finite distance and the stresses remain finite throughout—excepting, of course, at the points where the concentrated loads act.

We then obtain, as in § 15,

$$\left.\begin{aligned} U = {} & -\frac{W}{\pi}\int_0^\infty \frac{1}{u}\left(\frac{\frac{1}{\lambda'+\mu}\sinh u - \frac{1}{u}\,u\cosh u}{\sinh 2u + 2u}\right)\cos\frac{ul}{b}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\ & - \frac{Wy}{\mu\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\ & + \frac{W}{\pi}\int_0^\infty \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\cosh u - \frac{u}{\mu}\sinh u\right)}{\sinh 2u - 2u}\sin\frac{ul}{b}\sinh\frac{uy}{b}\cos\frac{ux}{b} - \left(\frac{1}{\lambda'+\mu}\right)\frac{3ly}{4u^2b^2}\right\}du \\ & + \frac{Wy}{\pi b\mu}\int_0^\infty \left(\frac{\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\cosh\frac{uy}{b}\cos\frac{ux}{b} - \frac{3}{4u^2}\frac{l}{b}\right)du \end{aligned}\right\} \quad (97)$$

$$
\left.\begin{aligned}
V = &-\frac{W}{\pi}\int_0^\infty \frac{1}{u}\left\{\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sinh u + \frac{u}{\mu}\cosh u}{\sinh 2u + 2u}\right\}\cos\frac{ul}{b}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{Wy}{\mu\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&-\frac{W}{\pi}\int_0^\infty\left[\frac{1}{u}\left\{\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u + \frac{u}{\mu}\sinh u}{\sinh 2u - 2u}\right\}\sin\frac{ul}{b}\cosh\frac{uy}{b}\sin\frac{ux}{b}\right. \\
&\qquad\qquad\left. - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4u^2}\frac{lx}{b^2}\right]du \\
&+\frac{Wy}{\mu\pi b}\int_0^\infty \frac{\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du
\end{aligned}\right\} \quad (97).
$$

$$
\left.\begin{aligned}
P = &-\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&-\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2W}{\pi b}\int_0^\infty \frac{\cosh u - u\sinh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \\
Q = &-\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2W}{\pi b}\int_0^\infty \frac{\cosh u + u\sinh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&+\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \\
S = &\ \frac{2W}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&-\frac{2W}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du
\end{aligned}\right\} \quad . \quad . \quad (98).
$$

Now, as before, these expressions may be expanded in powers of r about the origin. In this case they will be found to have a radius of convergence $\sqrt{l^2+b^2}$. Or they may be expanded about either point of concentrated loading, when they will have a radius of convergence $2\sqrt{l^2+b^2}$, or they may be split up as follows :—

Write

$$\frac{\cosh u}{\sinh 2u + 2u} = e^{-u} + e^{-u}\frac{(1 - 4u + e^{-2u})}{2(\sinh 2u + 2u)}$$

$$\frac{\cosh u}{\sinh 2u - 2u} = e^{-u} + e^{-u}\frac{(1 + 4u + e^{-2u})}{2(\sinh 2u - 2u)}$$

$$\frac{\sinh u}{\sinh 2u + 2u} = e^{-u} + e^{-u}\frac{(-1 - 4u + e^{-2u})}{2(\sinh 2u + 2u)}$$

$$\frac{\sinh u}{\sinh 2u - 2u} = e^{-u} + e^{-u}\frac{(-1 + 4u + e^{-2u})}{2(\sinh 2u - 2u)}$$

and consider separately the parts of the integrals due to the first and second terms of the right-hand sides of the above equations.

We find, after some reductions, on writing $b - y = y'$, $y'^2 + (x-l)^2 = r_1^2$, $y - b = y''$, $y''^2 + (x+l)^2 = r_2^2$, $(x+l)/y'' = \tan\phi_2$, $(x-l)/y' = \tan\phi_1$,

$$\left.\begin{aligned} P &= -\frac{W\cos\phi_1}{\pi r_1} - \frac{W\cos\phi_2}{\pi r_2} + \frac{Wy'}{\pi r_1^2}\cos 2\phi_1 + \frac{Wy''}{\pi r_2^2}\cos 2\phi_2 + P_2 \\ Q &= -\frac{W\cos\phi_1}{\pi r_1} - \frac{W\cos\phi_2}{\pi r_2} - \frac{Wy'}{\pi r_1^2}\cos 2\phi_1 - \frac{Wy''}{\pi r_2^2}\cos 2\phi_2 + Q_2 \\ S &= \frac{Wy'}{\pi r_1^2}\sin 2\phi_1 - \frac{Wy''}{\pi r_2^2}\sin 2\phi_2 + S_2 \end{aligned}\right\} \quad . \quad . \quad (99),$$

where

$$\left.\begin{aligned} P_2 = {} & \frac{W}{\pi b}\int_0^\infty \frac{(1 + 5u - 4u^2 - (1-u)\,e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ & - \frac{W}{\pi b}\int_0^\infty \frac{(1 + 5u - 4u^2 + (1-u)\,e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ & + \frac{Wy}{\pi b^2}\int_0^\infty \frac{u\,(1 + 4u - e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ & - \frac{Wy}{\pi b^2}\int_0^\infty \frac{u\,(1 + 4u + e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \\ Q_2 = {} & \frac{W}{\pi b}\int_0^\infty \frac{(1 + 3u + 4u^2 - (1+u)\,e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ & - \frac{W}{\pi b}\int_0^\infty \frac{(1 + 3u + 4u^2 + (1+u)\,e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ & - \frac{Wy}{\pi b^2}\int_0^\infty \frac{u\,(1 + 4u - e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ & + \frac{Wy}{\pi b^2}\int_0^\infty \frac{u\,(1 + 4u + e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \end{aligned}\right\} \quad . \quad (100),$$

$$\left.\begin{aligned} S_2 = {} & \frac{W}{\pi b}\int_0^\infty \frac{u\,(1 - 4u + e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ & - \frac{W}{\pi b}\int_0^\infty \frac{u\,(-1 + 4u + e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ & + \frac{Wy}{\pi b^2}\int_0^\infty \frac{u\,(1 + 4u - e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ & + \frac{Wy}{\pi b^2}\int_0^\infty \frac{u\,(1 + 4u + e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \end{aligned}\right\} \quad . \; (100).$$

P_2, Q_2, S_2 are finite and continuous all over the beam. They may be expanded in powers of r about the origin, the series being convergent inside a circle of radius $\sqrt{l^2 + (3b)^2}$, so that the points of concentrated loading are included. The parts of P, Q, S which become infinite at the points where the load acts are of the same form as if the beam were an infinite plate.

§ 29. *Series in Powers of r.*

We may here quote the expressions for P_2, Q_2, S_2 in powers of r. They are :

$$\begin{aligned} P_2 = {} & \frac{W}{\pi b}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}\,(1 + 5u - 4u^2 - (1 - u)\,e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du \\ & - \frac{W}{\pi b}\sum_1^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}\,(1 + 5u - 4u^2 + (1 - u)\,e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du \\ & + \frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!}\int_0^\infty \frac{u^{2\nu+2}\,(1 + 4u - e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du \\ & - \frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!}\int_0^\infty \frac{u^{2\nu+2}\,(1 + 4u + e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du, \end{aligned}$$

$$\begin{aligned} Q_2 = {} & \frac{W}{\pi b}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}\,(1 + 3u + 4u^2 - (1 + u)\,e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du \\ & - \frac{W}{\pi b}\sum_1^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}\,(1 + 3u + 4u^2 + (1 + u)\,e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du \\ & - \frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu + 1)!}\int_0^\infty \frac{u^{2\nu+2}\,(1 + 4u - e^{-2u})\,e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du \\ & + \frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu + 1)!}\int_0^\infty \frac{u^{2+2}\,(1 + 4u + e^{-2u})\,e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du, \end{aligned}$$

$$
\begin{aligned}
S_2 = {} & \frac{W}{\pi b}\sum_1^\infty \left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} \int_0^\infty \frac{u^{2\nu+1}(1-4u+e^{-2u})e^{-u}}{(\sinh 2u + 2u)} \cos\frac{ul}{b}\,du \\
& + \frac{W}{\pi b}\sum_0^\infty \left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} \int_0^\infty \frac{u^{2\nu+1}(1-4u-e^{-2u})e^{-u}}{(\sinh 2u - 2u)} \sin\frac{ul}{b}\,du \\
& + \frac{Wy}{\pi b^2}\sum_0^\infty \left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \int_0^\infty \frac{u^{2\nu+2}(1+4u-e^{-2u})e^{-u}}{\sinh 2u + 2u} \cos\frac{ul}{b}\,du \\
& + \frac{Wy}{\pi b^2}\sum_0^\infty \left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \int_0^\infty \frac{u^{2\nu+2}(1+4u+e^{-2u})e^{-u}}{\sinh 2u - 2u} \sin\frac{ul}{b}\,du,
\end{aligned}
$$

where $r^2 = x^2 + y^2$, $x = y \tan\phi$.

U, V may be broken up in like manner and the parts U_2, V_2 which remain finite and continuous everywhere can be expanded in the same way.

We shall require also the series for U, V, P, Q, S in powers of r, deduced directly from the expressions (98). They are

$$
\left.
\begin{aligned}
U = {} & -\frac{W}{\pi}\sum_0^\infty \left(\frac{1}{\lambda'+\mu}C_{2\nu-1} - \frac{1}{\mu}C_{2\nu}\right)\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin\widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{Wy}{\pi b\mu}\sum_1^\infty C_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!} + \frac{Wy}{\pi b\mu}\sum_0^\infty S_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!} \\
& + \frac{W}{\pi}\sum_0^\infty \left(\frac{1}{\lambda'+\mu}S_{2\nu-1} - \frac{1}{\mu}S_{2\nu}\right)\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos\widehat{2\nu+1}\,\phi}{(2\nu+1)!}, \\
V = {} & -\frac{W}{\pi}\sum_0^\infty \left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)C_{2\nu-1} + \frac{1}{\mu}C_{2\nu}\right\}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos\widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& + \frac{Wy}{\pi b\mu}\sum_0^\infty C_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!} + \frac{Wy}{\pi b\mu}\sum_1^\infty S_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!} \\
& - \frac{W}{\pi}\sum_0^\infty \left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)S_{2\nu-1} + \frac{1}{\mu}S_{2\nu}\right\}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin\widehat{2\nu+1}\,\phi}{(2\nu+1)!}, \\
P = {} & -\frac{2W}{\pi b}\sum_0^\infty (C_{2\nu-1} - C_{2\nu})\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!} - \frac{2Wy}{\pi b^2}\sum_0^\infty C_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos\widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{2W}{\pi b}\sum_1^\infty (S_{2\nu-1} - S_{2\nu})\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!} - \frac{2Wy}{\pi b^2}\sum_0^\infty S_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin\widehat{2\nu+1}\,\phi}{(2\nu+1)!}, \\
Q = {} & -\frac{2W}{\pi b}\sum_0^\infty (C_{2\nu-1} + C_{2\nu})\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!} + \frac{2Wy}{\pi b^2}\sum_0^\infty C_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos\widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{2W}{\pi b}\sum_1^\infty (S_{2\nu-1} + S_{2\nu})\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!} + \frac{2Wy}{\pi b^2}\sum_0^\infty S_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi}{(2\nu+1)!}, \\
S = {} & \frac{2W}{\pi b}\sum_1^\infty C_{2\nu}\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!} - \frac{2Wy}{\pi b^2}\sum_0^\infty C_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin\widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{2W}{\pi b}\sum_0^\infty S_{2\nu}\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!} + \frac{2Wy}{\pi b^2}\sum_0^\infty S_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos\widehat{2\nu+1}\,\phi}{(2\nu+1)!},
\end{aligned}
\right\} \quad (101),
$$

where

$$\left.\begin{aligned}
C_{2\nu} &= \int_0^\infty \frac{u^{2\nu+1}\cosh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du \qquad \nu = 0, 1, 2, \ldots \\
C_{2\nu+1} &= \int_0^\infty \frac{u^{2\nu+2}\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du \qquad \nu = -1, 0, 1, 2, \ldots \\
S_{2\nu} &= \int_0^\infty \frac{u^{2\nu+1}\sinh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du \qquad \nu = 0, 1, 2, \ldots \\
S_{2\nu+1} &= \int_0^\infty \frac{u^{2\nu+2}\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du \qquad \nu = 0, 1, 2, \ldots
\end{aligned}\right\} \quad . \quad . \quad (102).$$

and

$$S_{-1} = \int_0^\infty \left\{\frac{\cosh u \sin ul/b}{\sinh 2u - 2u} - \frac{3l}{4u^2 b}\right\} du.$$

§ 30. *Distortion of the Axis of the Beam.*

If in the expression for V we write $y = 0$ we obtain the equation of the distorted form of the axis

$$V = -\frac{W}{\pi}\sum_0^\infty\left(\left\{\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right\}S_{2\nu-1} + \frac{1}{\mu}S_{2\nu}\right)(-1)^\nu\left(\frac{x}{b}\right)^{2\nu+1}\frac{1}{(2\nu+1)!}.$$

To the first approximation it is a semi-cubical parabola

$$V = -\frac{W}{\pi}\left\{\left(\frac{4}{E}S_{-1} + \frac{1}{\mu}S_0\right)\frac{x}{b} - \left(\frac{4}{E}S_1 + \frac{1}{\mu}S_2\right)\frac{x^3}{6b^3}\right\}.$$

This holds if x be small compared with b. If further we have l small compared with b, so that the two concentrated loads are applied in near parallel lines (*e.g.*, as in the case of material pressed between the edges of a pair of scissors), then we have, to the first approximation,

$$\begin{aligned}
S_{-1} &= \frac{l}{b}\int_0^\infty\left(\frac{u\cosh u}{\sinh 2u - 2u} - \frac{3}{4u^2}\right)du - \frac{l^3}{6b^3}\int_0^\infty\frac{u^3\cosh u\,du}{\sinh 2u - 2u}, \\
&= \frac{l}{b}F_1 - \tfrac{1}{6}\frac{l^3}{b^3}F_3 \text{ (see p. 99)}, \\
S_0 &= \frac{l}{b}G_2 - \tfrac{1}{6}\frac{l^3}{b^3}G_4, \\
S_1 &= \frac{l}{b}F_3 - \tfrac{1}{6}\frac{l^3}{b^3}F_5, \\
S_2 &= \frac{l}{b}G_4 - \tfrac{1}{6}\frac{l^3}{b^3}G_6.
\end{aligned}$$

The terms of order l^3/b^3 may be dropped in the coefficient of x^3/b^3, the latter quantity being already small, and we have finally

$$V = -\frac{W}{\pi E}\frac{lx}{b^2}\left[\left(4F_1 + \frac{E}{\mu}G_2\right) - \left(\tfrac{4}{6}F_3 + \frac{E}{\mu}\frac{G_4}{6}\right)\frac{l^2 + x^2}{b^2}\right],$$

or putting $E = 5\mu/2$ to simplify the arithmetic

$$V = -\frac{Wx}{Eb^2}\left[2\cdot80 - 4\cdot96\frac{l^2 + x^2}{b^2}\right].$$

The slope of the strained form of the axis at the origin is therefore a maximum when $2\cdot80 - 4\cdot96 \times \frac{3l^2}{b^2} = 0$, or $l/b = \cdot434$.

For such a value of l/b the approximation will not be quite valid. Still, it will be sufficient, even then, to give a rough idea of the values of the coefficients.

Assuming the formula given for V to hold for this value of l/b, we see that this greatest slope is $-\frac{W}{Eb}(\cdot810)$.

Now if the part of the beam between $x = \pm l$ were subjected to a uniform shear $W/2b$ giving the same total shear across the section, then, if the sections $x = \pm l$ were kept vertical, we should have $V = -\frac{W}{2b}\frac{x}{\mu} = -\frac{W}{Eb}x \times 1\cdot25$, if $E = 5\mu/2$. This gives a slope nearly 3/2 of the preceding one.

§ 31. *Distortion of the Cross-section $x = 0$, and Shear in that Cross-section.*

If we work out in the same way the value of U for $x = 0$ we find

$$U = \frac{W}{\pi}\sum_0^\infty\left(\frac{y}{b}\right)^{2\nu+1}\left\{\left(\frac{1}{\lambda' + \mu}S_{2\nu-1} - \frac{1}{\mu}S_{2\nu}\right)\frac{1}{(2\nu+1)!} + \frac{1}{\mu}S_{2\nu-1}\frac{1}{(2\nu)!}\right\}.$$

If l be very small and y/b sufficiently small for 5th and higher powers to be neglected, this gives, assuming $E = 5\mu/2$ to simplify the arithmetic,

$$U = \frac{Wyl}{\pi Eb^2}\left[(4\,F_1 - 2\cdot5\,G_2) + \frac{y^2}{b^2}\left(\tfrac{3}{2}F_3 - \tfrac{5}{12}G_4\right)\right],$$

i.e.,

$$U = \frac{Wyl}{\pi Eb^2}\left[-5\cdot292 + \frac{y^2}{b^2}(\cdot492)\right].$$

We see, therefore, that the y^3 term is practically negligible, or, for a very large range of y, the mid-section remains sensibly plane.

For the shear in this cross-section, we have

$$S = -\frac{2W}{\pi b}\sum_0^\infty S_{2\nu}\left(\frac{y}{b}\right)^{2\nu}\frac{1}{(2\nu)!} + \frac{2W}{\pi b}\sum_0^\infty S_{2\nu+1}\left(\frac{y}{b}\right)^{2\nu+2}\frac{1}{(2\nu+1)!}$$

or

$$S = -\frac{2W}{\pi b}S_0 - \frac{2W}{\pi b}\frac{y^2}{b^2}\left(\left(\frac{S_2}{2}\right) - S_1\right) - \text{higher terms.}$$

S is therefore a numerical minimum at the centre if $\frac{S_2}{2} - S_1 > 0$.

Now for the small values of l/b

$$S_2 = (l/b)\,G_4 - \tfrac{1}{6}(l/b)^3\,G_6,$$

$$S_1 = (l/b)\,F_3 - \tfrac{1}{6}(l/b)^3\,F_5.$$

But since $G_4 = 24{\cdot}824$, $F_3 = 7{\cdot}224$, when l/b is small $S_2 > 2S_1$, and the shear increases from the centre outwards. This is shown by the full curve (a) in fig. vi.

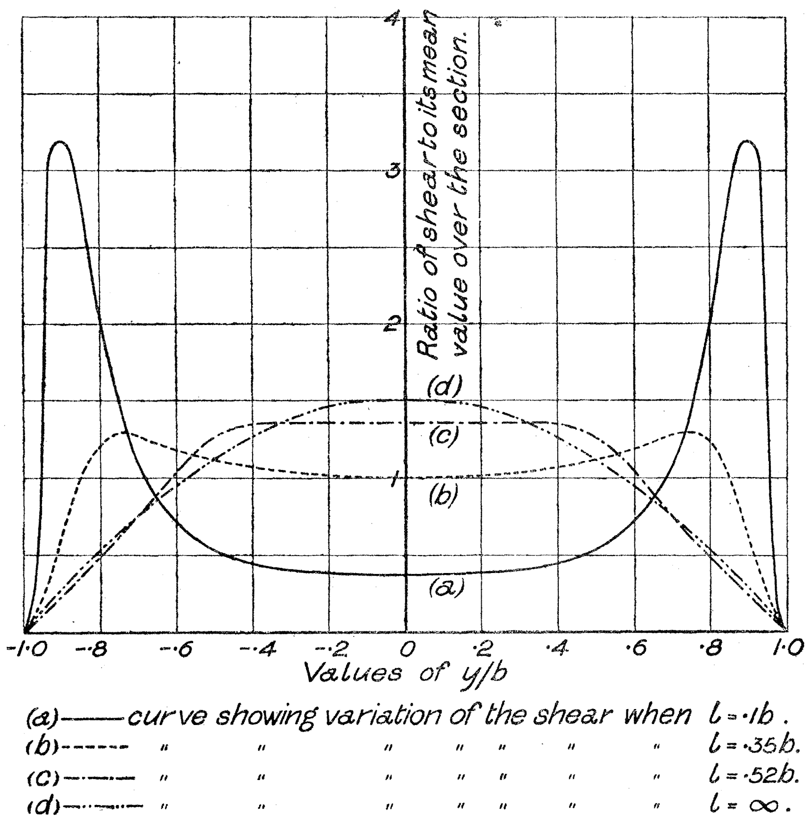

Fig. vi.

Near the edges $y = \pm b$, if l be small, the terms $\frac{Wy'}{\pi r_1^2}\sin 2\phi_1 - \frac{Wy''}{\pi r_2^2}\sin 2\phi_2$ will be the most important. Hence the shear is a minimum at the centre, increases to a high maximum corresponding to a distance from the edge equal to l approximately, and decreases down again to zero. The full curve in fig. vi. has been drawn for $l = b/10$.

As we increase l, these maxima at the sides become smaller and smaller and move towards the centre. At the same time the shear at the centre increases.

When l/b is made indefinitely large it is easily seen that S_0 and S_1 tend to the finite limit $3\pi/8$ whereas S_2 and all the others tend to zero.

Hence, for some value of l/b we must have $S_2 = 2S_1$.

It we calculate the values of S_2 and S_1 for $l/b = \pi/6, \pi/3, \pi/2$, we find

l/b.	S_1.	S_2.
$\pi/6$	1·9862	3·9475
$\pi/3$	1·2585	·3235
$\pi/2$	—	– ·0591

From these values and from the known behaviour of these functions near $l/b = 0$ and $l/b = \infty$ we can draw a rough diagram illustrating their variations. Fig. vii.

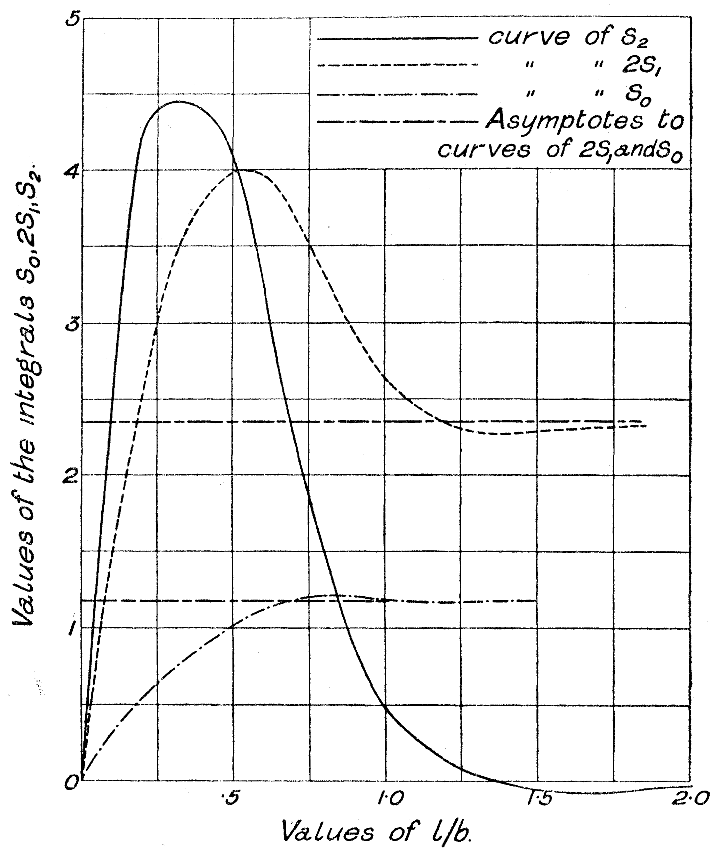

Fig. vii.

gives the curves of S_0, $2S_1$, and S_2 as we increase l. It will be seen from the figure that S_2 and $2S_1$ intersect when $l/b = ·52$ nearly.

Hence, when the arm of the couple is about half the height of the beam the shear is stationary at the centre, a horizontal straight line having contact of the third order with the curve. Curve (c), fig. vi., shows the distribution of shear, roughly sketched, for this case. It is easy to see that the centre corresponds to a maximum for the shear, for the next higher terms in the expansion of S are $-\frac{2W}{\pi b}\frac{y^4}{b^4}\left(\frac{S_4}{4!}-\frac{S_3}{3!}\right)$.

We have therefore a numerical maximum if $S_4 < 4S_3$, and a rough numerical calculation enables us to verify that this is the case.

The shear is therefore greatest at the centre, but decreases extremely slowly and remains constant over nearly half the section.

Another case of interest presents itself when the shear at the centre is exactly equal to its mean value over the section.

This occurs when $S_0 = \cdot 7854 = \pi/4$.

If we write $S_0 = (l/b)\,G_2 - \frac{1}{6}(l/b)^3\,G_4 = 2\cdot 818\ l/b - 4\cdot 138\ l^3/b^3$, we find that this roughly corresponds to $l/b = \cdot 32$.

Measured on the diagram for S_0 on fig. vii. the value of l/b corresponding to $S_0 = \pi/4$ would be about ·35. This latter value is probably the more correct, as for values of $l/b > \cdot 3$ the above approximation for S_0 is hardly sufficient.

In this case it is found that $S_2/2 - S_1 = \cdot 4$ roughly. The shear is therefore a minimum at the centre. It increases as we proceed outwards, but not very rapidly, and decreases down to zero at the edges. The curve is shown as (b) on fig. vi. The total area of the curve reckoned from a horizontal tangent at the middle point as base is zero, *i.e.*, there is as much above as below.

Finally, curve (d) on fig. vi. shows the distribution of shear when the arm of the couple is indefinitely increased. This is the parabola

$$S = -\frac{3W}{4b^3}\left(b^2 - y^2\right).$$

It is striking how very early this limiting distribution is reached. Fig. vii. already shows that the coefficients of the series reach their limiting values with great rapidity. For an arm of the couple equal to twice the height of the beam, the parabolic distribution of shear, corresponding to a long cantilever, will, at the mid-section, be practically undisturbed.

§ 32. *Practical Importance of this Problem.*

The problem which has been investigated in this part of the paper is one of considerable importance in practice. The only way in which we can apply a shearing force to materials is by means of two opposite asymmetrically situated pressures, such as we have dealt with in this case. The case of material cut through by scissors, which is frequently quoted as an example of the application of shearing stress, really corresponds to a stress-distribution of this kind. Similarly, a rivet which fastens together two plates is subjected to stress-systems of this type whenever the compound plate undergoes strain in its own plane. In nearly every modern engineering structure, such as railway bridges, &c., cases of this kind are of constant occurrence, and the strength of the structure depends, to a very great extent, upon the strength of the individual rivets. It becomes therefore a problem of the very greatest practical importance to know how the distribution of shear inside such a rivet varies with the dimensions of the rivet and with the thickness of the plates. At present our knowledge of the subject is purely empirical; and although the results of the present paper apply only to a rivet of rectangular section, and even then are only an approximation, yet they should furnish some indications which may be of value in other cases.

Another point which is illustrated by these results is the manner in which DE SAINT-VENANT'S solutions are modified, when we gradually bring the terminal systems of load closer together. We see that the modifications introduced are practically insensible at distances from the section where the load is applied which are greater than the height of the beam. This is of importance, as it tells us within which limits, in any experiment, we may assume the state of a beam to be given by one of the "uniform" solutions which only depend upon the total terminal conditions and which are transmitted without change of type.

PART IV.

SOLUTION FOR A BEAM WHOSE UPPER AND LOWER BOUNDARIES ARE ACTED UPON BY SHEARING STRESS ONLY.

§ 33. *Expressions for the Displacements and Stresses in Series and Integrals.*

Let us now consider a beam acted upon by shearing stress alone, over the boundaries $y = \pm b$. Then, in the general solution of § 7, $\alpha_n = \beta_n = \gamma_n = \delta_n = 0$.

If further we suppose the shear to reduce to a single concentrated force L at one point $(0, b)$ we have $\zeta_0 = \frac{L}{2a}$, $\zeta_n = \frac{L}{a}$, $\kappa_n = 0 = \theta_n = \nu_n$.

Putting in these values into (44), (45), (46), (47), (48), (54), and (55) we obtain

$$\left.\begin{aligned} U = & -\frac{\lambda' + 2\mu}{32\mu(\lambda' + \mu)}\frac{Lx^2}{ab} + \frac{3\lambda' + 2\mu}{32\mu(\lambda' + \mu)}\frac{Ly^2}{ab} + \frac{1}{\mu}\frac{Ly}{8a} - Ay + B \\ & + \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\cosh mb - \frac{1}{\mu}mb\sinh mb}{\sinh 2mb + 2mb}\cosh my\cos mx \\ & + \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\sinh mb - \frac{1}{\mu}mb\cosh mb}{\sinh 2mb - 2mb}\sinh my\cos mx \\ & + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\cosh mb\sinh my}{\sinh 2mb + 2mb}\cos mx + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\sinh mb\cosh my}{\sinh 2mb - 2mb}\cos mx, \end{aligned}\right\} \quad (103)$$

$$\left.\begin{aligned}
V = {} & \frac{\lambda'}{16\mu(\lambda'+\mu)}\frac{Lxy}{ab} - \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\frac{1}{\lambda'+\mu}\cosh mb + \frac{1}{\mu}mb\sinh mb}{\sinh 2mb + 2mb}\sinh my \sin mx \\
& - \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\frac{1}{\lambda'+\mu}\sinh mb + \frac{1}{\mu}mb\cosh mb}{\sinh 2mb - 2mb}\cosh my \sin mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\cosh mb\cosh my}{\sinh 2mb + 2mb}\sin mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\sinh mb\sinh my}{\sinh 2mb - 2mb}\sin mx + Ax + C, \\
P = {} & -\frac{Lx}{4ab} - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{4\cosh mb - 2mb\sinh mb}{\sinh 2mb + 2mb}\cosh my\sin mx \\
& - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{4\sinh mb - 2mb\cosh mb}{\sinh 2mb - 2mb}\sinh my\sin mx \\
& - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\cosh mb\sinh my}{\sinh 2mb + 2mb}\sin mx - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\sinh mb\cosh my}{\sinh 2mb - 2mb}\sin mx, \\
Q = {} & -\sum_{n=1}^{\infty}\frac{L}{2a}\frac{2mb\sinh mb\cosh my}{\sinh 2mb + 2mb}\sin mx - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2mb\cosh mb\sinh my}{\sinh 2mb - 2mb}\sin mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\cosh mb\sinh my}{\sinh 2mb + 2mb}\sin mx + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\sinh mb\cosh my}{\sinh 2mb - 2mb}\sin mx, \\
S = {} & \frac{Ly}{4ab} + \frac{L}{4a} + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\cosh mb\cosh my}{\sinh 2mb + 2mb}\cos mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\sinh mb\sinh my}{\sinh 2mb - 2mb}\cos mx \\
& + \sum_{n=1}^{\infty}\frac{L}{a}\frac{\cosh mb - mb\sinh mb}{\sinh 2mb + 2mb}\sinh my\cos mx \\
& + \sum_{n=1}^{\infty}\frac{L}{a}\frac{\sinh mb - mb\cosh mb}{\sinh 2mb - 2mb}\cosh my\cos mx
\end{aligned}\right\} \quad (103),$$

where $m = n\pi/a$, and A, B, C are arbitrary constants to be determined from the fixing conditions.

Now if the fixing conditions are

(i.) That the displacement of the origin is to be zero;

(ii.) That the extremities of the axis are to remain on the same horizontal line, then

$$C = 0,$$

$$B = -\sum_{n=1}^{\infty}\frac{L}{2am}\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh mb - \frac{1}{\mu}mb\sinh mb}{\sinh 2mb + 2mb},$$

$$A = 0;$$

but if we put in these values and then proceed to make a infinite, certain parts of the expressions for U and V do not give finite integrals in the limit.

This is due to the fact that the conditions of rigid equilibrium require shears $Lb/2a$ at the two ends (fig. viii.).

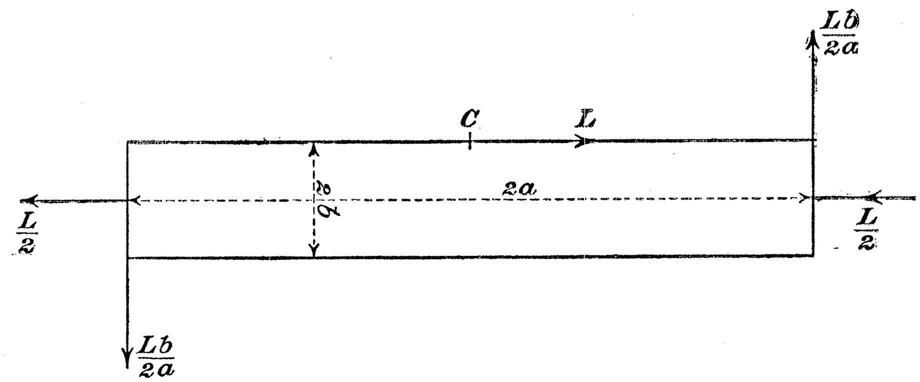

Fig. viii.

These shears $Lb/2a$ will produce a deflection due to bending alone, which, calculated from the Euler-Bernoulli formula, comes to

$$V = \frac{3L}{32ab^2}\left(ax^2 - \frac{x^3}{3}\right)\left(\frac{1}{\mu} + \frac{1}{\lambda' + \mu}\right) \quad (\text{for } x > 0),$$

and when a is made very large, this gives

$$V = \frac{3L}{32}\frac{x^2}{b^2}\left(\frac{1}{\mu} + \frac{1}{\lambda' + \mu}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

for the bending deflection produced by the end shears at large distances x, which, however, are still finite compared with a. If, therefore, we allow the beam to bend freely under these end loads, in such a way that each of these produces its proper bending deflection and no more, the constant A must be adjusted so that, for large values of x, V tends to the value (104).

This implies that A must have an infinite part, which will exactly cancel the infinite part of V. It is easily found that the value

$$A = \left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\frac{3}{8}\frac{L}{a}\sum_{n=1}^{\infty}\frac{1}{m^2b^2} + A',$$

where A' is finite, will introduce terms in both U and V which will make these quantities remain finite in the limit when a is infinite.

We then find, putting in for B the value found and proceeding to the limit,

$$\left.\begin{aligned}
\mathrm{U} = {} & \frac{\mathrm{L}}{2\pi}\int_0^\infty \frac{1}{u}\left[\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u - \frac{u}{\mu}\sinh u}{\sinh 2u + 2u}\left(\cosh\frac{uy}{b}\cos\frac{ux}{b} - 1\right)\right] du \\
& + \frac{\mathrm{L}}{2\pi}\int_0^\infty \frac{1}{u}\left[\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sinh u - \frac{u}{\mu}\cosh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b} - \frac{1}{\lambda'+\mu}\frac{3y}{4ub}\right] du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\frac{\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\, du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\left[\frac{\sinh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b} - \frac{3}{4u^2}\right] du - \mathrm{A}'y, \\
\mathrm{V} = {} & -\frac{\mathrm{L}}{2\pi}\int_0^\infty \frac{1}{u}\left[\frac{\frac{1}{\lambda'+\mu}\cosh u + \frac{1}{\mu}u\sinh u}{\sinh 2u + 2u}\right]\sinh\frac{uy}{b}\sin\frac{ux}{b}\, du \\
& - \frac{\mathrm{L}}{2\pi}\int_0^\infty \left[\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3x}{4u^2 b}\right] du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\frac{\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\, du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\frac{\sinh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\, du + \mathrm{A}'x
\end{aligned}\right\} \quad (105).$$

Now, when in V we put $y = 0$, we have left

$$-\frac{\mathrm{L}}{2\pi}\int_0^\infty \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u}\sin\frac{ux}{b} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3x}{4u^2 b}\right\} du + \mathrm{A}'x.$$

This integral may be written as the sum of two others,

$$-\frac{\mathrm{L}}{2\pi}\int_0^\infty \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4u^3} - \frac{1}{40u}\left(\frac{9}{\mu} - \frac{1}{\lambda'+\mu}\right)\right\}\sin\frac{ux}{b}\, du$$

$$+\frac{\mathrm{L}}{2\pi}\int_0^\infty \left\{\left[\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4}\frac{ux/b - \sin ux/b}{u^3}\right] - \frac{1}{40u}\left(\frac{9}{\mu} - \frac{1}{\lambda'+\mu}\right)\sin\frac{ux}{b}\right\} du.$$

Consider the first of these integrals, and let

$$f(u) = \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4u^3} - \frac{1}{40u}\left(\frac{9}{\mu} - \frac{1}{\lambda'+\mu}\right)\right\}.$$

Then $f(u)$ and its differential coefficients are finite and continuous for all values of u, and vanish for $u = \infty$. $f(u)$ itself $= 0$, when $u = 0$ and the integral $\int_0^\infty |f''(u)|\, du$ is finite, $|f'(u)|$ denoting the absolute value of $f'(u)$. It is then easy to see that $I = \int_0^\infty f(u) \sin u\xi \, du$ tends to zero as ξ tends to infinity. For, integrating by parts

$$I = \left[-\frac{1}{\xi} \cos u\xi f(u) \right]_0^\infty + \frac{1}{\xi} \int_0^\infty \cos u\xi f'(u) \, du$$

$$= \frac{1}{\xi} \int_0^\infty \cos u\xi f'(u) \, du.$$

But $\left| \int_0^\infty \cos u\xi f'(u) \, du \right| < \int_0^\infty |f'(u)| \, du <$ a finite quantity M; hence $I < \frac{M}{\xi}$,

and therefore tends to zero as ξ tends to infinity.

Hence, when x is large, V reduces to the second integral. The latter can be evaluated, and it comes to

$$\frac{3}{32} L \left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu} \right) \frac{x^2}{b^2} - \frac{L}{160} \left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu} \right)$$

for $x > 0$ and

$$-\frac{3L}{32} \left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu} \right) \frac{x^2}{b^2} + \frac{L}{160} \left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu} \right) \text{ for } x < 0.$$

The first terms correspond to the bending due to the shears at the ends.

We should therefore try to make $A'x - \frac{L}{160} \left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu} \right) = 0$ for all large values of x.

This is obviously impossible. But $A'x$ being eventually the most important term, the condition is approximately fulfilled by taking $A' = 0$. This determines U and V. We see that the effect of the isolated shear L is to *deflect* the central line of the beam through the distance $2 \times \frac{L}{160} \left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu} \right)$ *away* from its line of action.

Putting $A' = 0$ in equations (105) they give us U and V. Integral expressions for the stresses are obtained in like manner. They are

$$\left. \begin{aligned} P = &- \frac{L}{\pi b} \int_0^\infty \frac{2 \cosh u - u \sinh u}{\sinh 2u + 2u} \cosh \frac{uy}{b} \sin \frac{ux}{b} \, du \\ &- \frac{L}{\pi b} \int_0^\infty \frac{2 \sinh u - u \cosh u}{\sinh 2u - 2u} \sinh \frac{uy}{b} \sin \frac{ux}{b} \, du \\ &- \frac{L}{\pi b} \int_0^\infty \frac{uy}{b} \frac{\cosh u}{\sinh 2u + 2u} \sinh \frac{uy}{b} \sin \frac{ux}{b} \, du \\ &- \frac{L}{\pi b} \int_0^\infty \frac{uy}{b} \frac{\sinh u}{\sinh 2u - 2u} \cosh \frac{uy}{b} \sin \frac{ux}{b} \, du \end{aligned} \right\} \quad . \quad . \quad . \quad (106)$$

$$\left.\begin{aligned}
Q = &-\frac{L}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&-\frac{L}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&+\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&+\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du, \\
S = &\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{L}{\pi b}\int_0^\infty \frac{\cosh u - u\sinh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{L}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du
\end{aligned}\right\} \quad . \quad . \quad (106).$$

§ 34. *Expressions for the Displacements and Stresses in Series of Powers of the Radius Vector from a Point.*

The expressions given above for U, V, P, Q, S may be transformed exactly as in §§ 16, 17, and we obtain expansions about the point $(0, b)$ where the shear is applied. Eventually, r', ϕ' having the same meaning as on p. 92, we find:

$$\left.\begin{aligned}
U = &-\frac{L}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\log\left(\frac{r'}{b}\right) - \frac{L}{2\pi\mu}\frac{y'}{r'}\cos\phi' \\
&+\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left\{D + \sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+2}\frac{\cos(2\nu+2)\phi'}{(2\nu+2)!}(H_{2\nu+1}-H_{2\nu})\right\} \\
&-\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu} \\
&-\frac{2Ly'}{\pi b}\frac{1}{\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}H_{2\nu} + \frac{2Ly'}{\pi b}\frac{1}{\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1}-H_{2\nu}), \\
V = &-\frac{L\phi'}{2\pi(\lambda'+\mu)} - \frac{L}{2\pi\mu}\frac{y'}{r'}\sin\phi' \\
&-\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu} \\
&\qquad+\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}(H_{2\nu-1}-H_{2\nu-2}) \\
&-\frac{2Ly'}{\pi b\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1}-H_{2\nu}) + \frac{2Ly'}{\pi b\mu}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu},
\end{aligned}\right\}(107),$$

$$\left.\begin{aligned}
P = &-\frac{2L}{\pi}\frac{\sin\phi'}{r'} + \frac{L}{\pi}\frac{y'}{r'^2}\sin 2\phi' \\
&- \frac{8L}{\pi b}\sum_0^\infty \left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1} - H_{2\nu}) \\
&+ \frac{4L}{\pi b}\sum_1^\infty \left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu} + \frac{4Ly'}{\pi b^2}\sum_0^\infty \left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu+2} \\
&- \frac{4Ly'}{\pi b^2}\sum_1^\infty \left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}(H_{2\nu+1} - H_{2\nu}) \\
Q = &-\frac{L}{\pi}\frac{y'}{r'^2}\sin 2\phi' + \frac{4L}{\pi b}\sum_1^\infty \left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu} \\
&- \frac{4Ly'}{\pi b^2}\sum_0^\infty \left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu+2} + \frac{4Ly'}{\pi b^2}\sum_1^\infty \left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}(H_{2\nu+1} - H_{2\nu}) \\
S = &\frac{L}{\pi}\frac{\cos\phi'}{r'} - \frac{L}{\pi}\frac{y'\cos 2\phi'}{r'^2} - \frac{4L}{\pi b}\sum_0^\infty \left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1} - H_{2\nu}) \\
&- \frac{4Ly'}{\pi b^2}\sum_0^\infty \left(\frac{r'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}(H_{2\nu+1} - H_{2\nu}) + \frac{4Ly'}{\pi b^2}\sum_0^\infty \left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu+2}
\end{aligned}\right\} (107),$$

where the H's are given by equations (80) and are the same as before; and

$$D = \int_0^\infty \left(\frac{u - \frac{1}{2} + \frac{1}{8}u^{-1} - \frac{1}{8}u^{-1}e^{-4u}}{\sinh^2 2u - 4u^2} + \frac{e^{-u}}{4u} - \frac{3}{16u^2} - \frac{1}{4u}\frac{\cosh u}{\sinh 2u + 2u}\right)du$$
$$+ \frac{E}{4\mu}\int_0^\infty \frac{\sinh u}{4(\sinh 2u + 2u)}\,du.$$

The leading terms in U, V, P, Q, S which precede the Σ's form what is left of this solution when b is made infinite. They give therefore the displacements and stresses due to a shear acting at an edge of an infinite plate.

They will be found to agree with the expressions obtained by BOUSSINESQ ('Comptes Rendus,' vol. 114, pp. 1465–1468) for an infinite solid, the strain being two-dimensional; provided that λ be changed into λ'.

At the point of loading itself the stresses are infinite and the displacements infinite or indeterminate.

The series in the expressions (107) are easily seen to have a radius of convergence $4b$.

The series for the shear reveals a very curious phenomenon. The terms due to the infinite plate may be written $\frac{2L}{\pi}\frac{x^2y'}{r'^4}$. They give therefore a *positive* shear throughout, and zero shear on the axis of y. But when the other terms are taken into account, the shear at points on the axis of y is

$$S = -\frac{4L}{\pi b}\left\{\frac{y'}{b}\,2(H_1 - H_0) - \frac{y'^2}{b^2}H_2 + \frac{y'^3}{b^3}\,\tfrac{2}{3}(H_3 - H_2) - \ldots\right\}$$
$$= -\frac{4L}{\pi b}\left\{\cdot 3638\,\frac{y'}{b} - \cdot 2271\,\frac{y'^2}{b^2} + \cdot 0733\,\frac{y'^3}{b^3} - \ldots\right\},$$

which gives a *negative* shear on the axis of y, as soon as we get away from the point of loading.

It follows that there must be, on either side of the cross-section through the load, a locus of points of zero shear.

It is easy to find the approximate form of this locus in the neighbourhood of the point of loading. Retaining only the leading terms in the Σ's in the expression for the shear, we find that $S = 0$ when

$$\frac{2L}{\pi}\frac{x^2 y'}{r'^4} = \frac{4L}{\pi b}\frac{y'}{b} \times 2(H_1 - H_0),$$

or

$$4(H_1 - H_0) r'^4 = x^2 b^2, \quad i.e., \quad r'^2 \mp \frac{bx}{2\sqrt{H_1 - H_0}} = 0.$$

These are two circles passing through the point of loading and having their centres lying on the upper edge of the beam, at a distance from the point of loading equal to $\frac{b}{4\sqrt{H_1 - H_0}} = \cdot 587b$. These give a kind of wedge-shaped area, similar to that

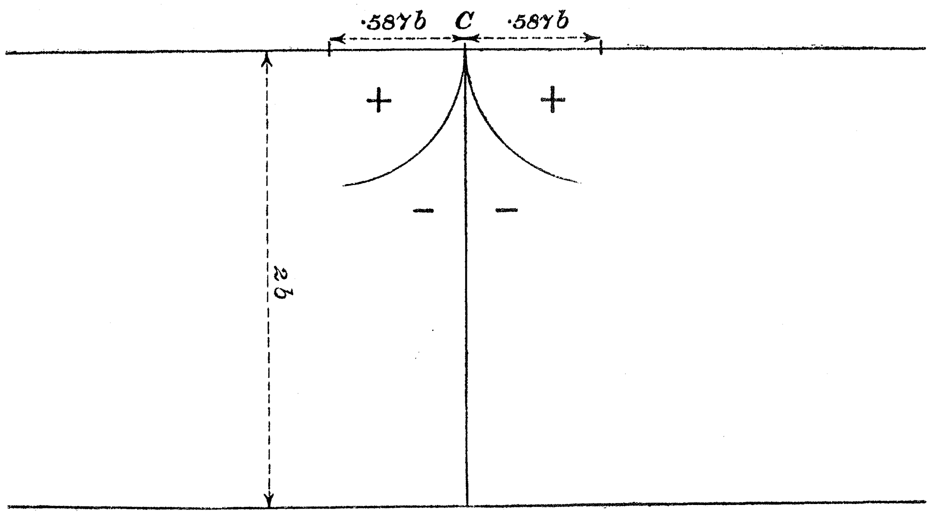

Fig. ix.

enclosed by the cusp of a caustic curve, inside which the shear is negative. This cusp is shown in fig. ix.

For higher values of y'/b this approximation will no longer hold, and the curve will deviate from the circle.

§ 35. *Distortion of the Beam.*

An interesting feature of a stress-system of this type is the distortion suffered by lines parallel to the axis of the beam.

We have already seen that at a certain distance the axis itself suffered a bodily shift, being depressed in front of the acting load and raised behind it.

The series for V in the neighbourhood of the load shows a similar phenomenon, points to the right of $x = 0$ being depressed by $\frac{L}{4}\frac{1}{\lambda' + \mu}$, and points to the left raised by the same amount.

$$(V)_{\substack{y'=0\\x>0}} = -\frac{L}{4(\lambda' + \mu)} - \frac{2L}{\pi}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\sum_0^\infty \left(\frac{x}{b}\right)^{2\nu+1} \frac{(-1)^\nu}{(2\nu+1)!} H_{2\nu},$$

H_0 being negative and H_2 being positive, as we go away from the load, the effect of the series is to decrease this effect, the level of the points in front of and behind the load tending to equalize itself. If we work out the series for V in the neighbourhood of the origin and of the point $(0, -b)$, we find (V) in the neighbourhood of origin

$$= -\frac{L}{2\pi}\sum_1^\infty \left(\frac{r}{b}\right)^\nu \frac{\sin\nu\phi}{\nu!}\left(\frac{G_{\nu-1}}{\lambda' + \mu} + \frac{F_\nu}{\mu}\right) + \frac{Ly}{2\pi b\mu}\sum_1^\infty \left(\frac{r}{b}\right)^\nu \frac{\sin\nu\phi}{\nu!} G_\nu,$$

where the F's and G's have the values given on p. 99, except that now

$$G_0 = \int_0^\infty \left(\frac{\sinh u}{\sinh 2u - 2u} - \frac{3}{4u^2}\right) du = -\cdot 2875.$$

Similarly (V) in neighbourhood of point $(0, -b)$

$$= -\frac{2L}{\pi}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\sum_0^\infty \left(\frac{r''}{b}\right)^{2\nu+1} \frac{\sin(2\nu+1)\phi''}{(2\nu+1)!} H'_{2\nu+1}$$

$$+ \frac{2L}{\pi}\left(\frac{1}{\lambda' + \mu}\right)\sum_1^\infty \left(\frac{r''}{b}\right)^{2\nu} \frac{\sin 2\nu\phi''}{(2\nu)!}(H'_{2\nu} - H'_{2\nu-1})$$

$$- \frac{2Ly''}{\pi b\mu}\sum_0^\infty \left(\frac{r''}{b}\right)^{2\nu+1} \frac{\sin(2\nu+1)\phi''}{(2\nu+1)!}(H'_{2\nu+2} - H'_{2\nu+1})$$

$$+ \frac{2Ly''}{\pi b\mu}\sum_1^\infty \left(\frac{r''}{b}\right)^{2\nu} \frac{\sin 2\nu\phi''}{(2\nu)!} H'_{2\nu+1},$$

where the H′'s have the value given on page 102.

From these expressions we obtain the following values for the transverse displacements of points on the lines $y = 0$, $y = -b$:—

$$V_0 = -\frac{L}{2\pi}\sum_0^\infty \left(\frac{x}{b}\right)^{2\nu+1} \frac{(-1)^\nu}{(2\nu+1)!}\left(\frac{1}{\lambda' + \mu} G_{2\nu} + \frac{1}{\mu} F_{2\nu+1}\right)$$

$$V_{-b} = -\frac{2L}{\pi}\sum_0^\infty \left(\frac{x}{b}\right)^{2\nu+1} \frac{(-1)^\nu}{(2\nu+1)!} H'_{2\nu+1}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right).$$

So that, approximately, putting in the values of the constants,

$$V_{+b} = \frac{L}{E}\left\{\begin{matrix} -\cdot 375\ (x+) \\ +\cdot 375\ (x-) \end{matrix} + \frac{x}{b}(\cdot 615) + \frac{x^3}{b^3}(\cdot 096) - \ldots\right\}$$

$$V_0 = \frac{L}{E}\left\{-\frac{x}{b}(\cdot 106) + \frac{x^3}{b^3}(\cdot 591) - \ldots\right\}$$

$$V_{-b} = \frac{L}{E}\left\{\frac{x}{b}(\cdot 125) + \frac{x^3}{b^3}(\cdot 228) - \frac{x^5}{b^5}(\cdot 041) + \ldots\right\},$$

where in the above uni-constant isotropy has been assumed to simplify the calculations, so that $\lambda' = \frac{2}{3}\mu$, $E = \frac{5}{2}\mu$.

The distortion, calculated from these formulæ, is represented on fig. x. for a range of x between $\pm \cdot 5b$. Curve (a) shows the distorted form of the upper edge, (b) that

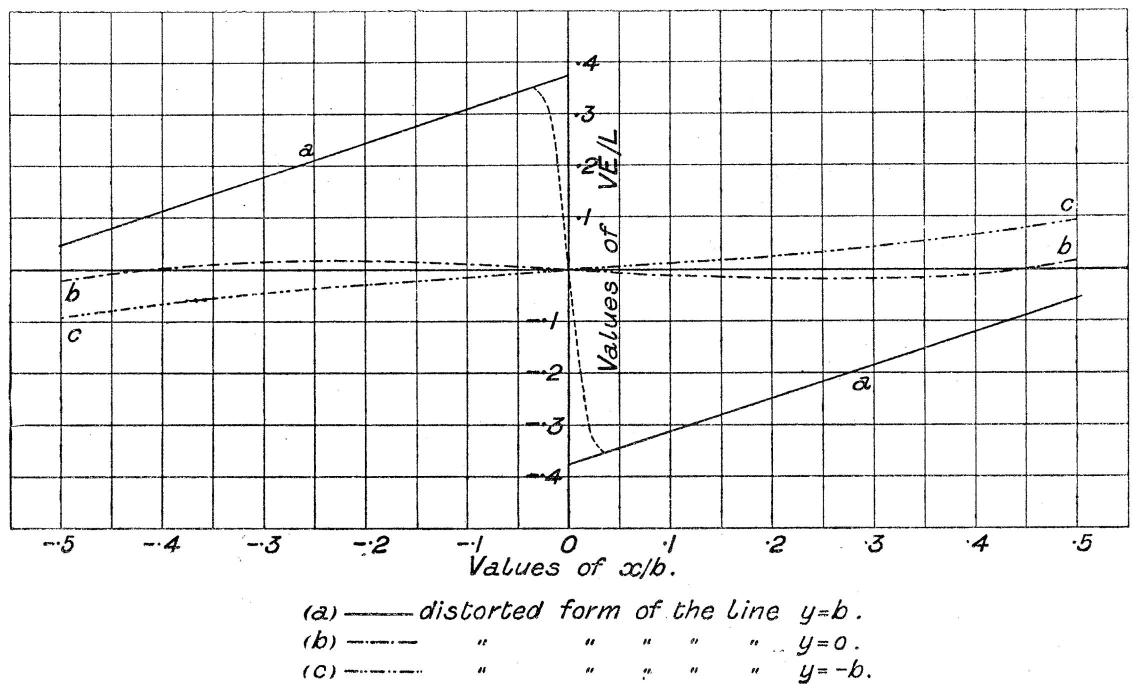

Fig. x.

of the axis, (c) that of the lower edge. With regard to (a) the limiting case, in which V is actually discontinuous, does not occur in practice. In order to get a real case, we have to take a horizontal line whose y' is very small without being actually zero. The discontinuity is then replaced by a very rapid variation, as shown by the dotted line.

The curves show that the depression produced in front of the load diminishes rapidly as we go away from the upper edge, and is even changed to a rise at the bottom of the beam. In every case, as we go away from the mid-section, the distorted lines rise to the right and fall to the left.

§ 36. *Case where the Shear is spread over an Area instead of a Line.*

As in § 22, we may consider the effect of distributing the concentrated shear over an area, instead of over a line. This is all the more important because, although we can, in practice, approximate to a line-distribution of pressure by means of a knife-edge, we cannot in the same way approximate to a line distribution of shear—shear being usually transmitted by means of projecting collars, which have a certain thickness. It is true that a thin notch might be cut into the material and an edge inserted in it which might be pulled sideways. But the cutting of such a notch would seriously weaken the material, besides altering the conditions so much as to render our solution inapplicable.

If we suppose our shear spread over a length $2a'$ of the upper edge, and if we adhere to the notation on p. 104, we find easily, L now denoting shearing force per unit area :—

$$\begin{aligned} U = &-\frac{L}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left\{(x+a')\log\frac{r_1}{b}-(x-a')\log\frac{r_2}{b}-2a'+y'(\phi_1-\phi_2)\right\} \\ &-\frac{Ly'}{2\pi\mu}(\phi_1-\phi_2)-\frac{2Lb}{\pi(\lambda'+\mu)}\sum_0^\infty H_{2\nu}\frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1-r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{(2\nu+2)!\,b^{2\nu+2}} \\ &+\frac{2Lb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left\{\frac{D}{b}\,.\,2a'+\sum_0^\infty\frac{r_1^{2\nu+3}\sin(2\nu+3)\phi_1-r_2^{2\nu+3}\sin(2\nu+3)\phi_2}{b^{2\nu+3}(2\nu+3)!}(H_{2\nu+1}-H_{2\nu})\right\} \\ &-\frac{2Ly'}{\pi\mu}\sum_0^\infty\frac{r_1^{2\nu+1}\sin(2\nu+1)\phi_1-r_2^{2\nu+1}\sin(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu} \\ &+\frac{2Ly'}{\pi\mu}\sum_0^\infty\frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1-r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1}-H_{2\nu}) \end{aligned}$$

$$\begin{aligned} V = &-\frac{L}{2\pi}\frac{1}{\lambda'+\mu}\left\{(x+a')\phi_1-(x-a')\phi_2-y'\log\frac{r_1}{r_2}\right\}-\frac{Ly'}{2\pi\mu}\log\frac{r_1}{r_2} \\ &+\frac{2Lb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty\frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1-r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu} \\ &-\frac{2Lb}{\pi(\lambda'+\mu)}\sum_1^\infty\frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1-r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu-1}-H_{2\nu-2}) \\ &+\frac{2Ly'}{\pi\mu}\sum_0^\infty\frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1-r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1}-H_{2\nu}) \\ &-\frac{2Ly'}{\pi\mu}\sum_1^\infty\frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1-r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu} \end{aligned}$$

$$P = -\frac{2L}{\pi}\log\frac{r_1}{r_2} - \frac{Ly'}{\pi}\left(\frac{\cos\phi_1}{r_1} - \frac{\cos\phi_2}{r_2}\right)$$
$$+ \frac{8L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1 - r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1} - H_{2\nu})$$
$$- \frac{4L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu}$$
$$- \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1 - r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu+2}$$
$$+ \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu+1} - H_{2\nu}).$$

$$Q = \frac{Ly'}{\pi}\left(\frac{\cos\phi_1}{r_1} - \frac{\cos\phi_2}{r_2}\right) - \frac{4L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu}$$
$$+ \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1 - r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu+2}$$
$$- \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu+1} - H_{2\nu}).$$

$$S = \frac{L}{\pi}(\phi_1 - \phi_2) - \frac{Ly'}{\pi}\left(\frac{\sin\phi_1}{r_1} - \frac{\sin\phi_2}{r_2}\right)$$
$$- \frac{4L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1 - r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1} - H_{2\nu})$$
$$- \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+1}\sin(2\nu+1)\phi_1 - r_2^{2\nu+1}\sin(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu+1} - H_{2\nu})$$
$$+ \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1 - r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu+2}.$$

The same remarks which were made on p. 106 as to the validity of such expressions apply here. Assuming that $2a' < 4b$, we may apply these to obtain the state of things near the layer of shear and at its extremities.

Clearly the only terms where discontinuities in U, V, P, Q, S, or their differential coefficients, may be introduced are their leading terms. Let us therefore study these.

It is easily seen that $(x + a')\log r_1$ and $(x - a')\log r_2$ are finite, continuous, and one-valued throughout, tending to 0 at the points $(\mp a', 0)$. Their differential coefficients with regard to y' are likewise everywhere finite, but are indeterminate at $(\mp a', 0)$. They introduce, however, no discontinuity if we proceed along $y' = 0$.

Similarly $y'\log r_1$ and $y'\log r_2$ are everywhere continuous, finite, and one-valued, and their differential coefficients with regard to x give no discontinuity if we keep to $y' = 0$.

$y'(\phi_1 - \phi_2)$ is everywhere continuous. Its differential coefficient with regard to y is indeterminate at $(\pm a', 0)$; if we proceed along $y' = 0$ it increases by π as we pass the point $(-a', 0)$ and decreases by π as we pass the point $(+a', 0)$. The same holds with regard to $(x + a')\phi_1 - (x - a')\phi_2$ and its differential coefficient with regard to x.

Hence, as far as U and V are concerned, they are both finite, continuous, and one-valued throughout the beam. $\frac{dU}{dy'}, \frac{dV}{dx}$ are everywhere finite, but are indeterminate at $(\pm a', 0)$. As we proceed along $y' = 0$, $\frac{dU}{dy'}$ decreases abruptly by $\frac{L}{2}\left(\frac{1}{\lambda' + \mu} + \frac{2}{\mu}\right)$ as we pass $(-a', 0)$ and increases again by the same amount as we pass $(+a', 0)$. Similarly $\frac{dV}{dx}$ decreases by $\frac{L}{2}\frac{1}{\lambda' + \mu}$ as we pass $(-a', 0)$, and increases by the same amount as we pass $(+a', 0)$. The first of these results means an abrupt change in the angle at which the distorted cross-sections meet the horizontal, and the second shows that the distorted form of the upper edge of the beam receives a sudden inflection downwards as we enter the layer of shear, and is again suddenly inflected upwards as we emerge from it.

It has been shown in a paper by the author "On the Equilibrium of Circular Cylinders under Certain Practical Systems of Load" ('Phil. Trans.,' A, vol. 198, pp. 147–233), that a precisely similar occurrence takes place in a circular cylinder subjected to a uniform ring of shear, over a certain length of its curved surface. The law that shear depresses the parts of the surface towards which it acts appears to be a general one.

Passing on now to consider the stresses P, Q, S, we find that Q and S remain everywhere finite, but are indeterminate at the points $(\pm a', 0)$. If we keep to $y' = 0$, Q is continuously zero and S changes by L at $(\pm a', 0)$, as it should. But P not only contains a part which becomes indeterminate at $(\pm a', 0)$, it also contains a term $-\frac{2L}{\pi}\log\frac{r_1}{r_2}$ which becomes infinite at those points.

This is a result for which we had no analogue in the case of a uniform layer of *pressure*. In that problem the stresses were everywhere finite. We now see that any finite discontinuity in the shear introduces an infinite pressure or tension P in the neighbourhood of this discontinuity. This result, again, has been found to hold good for circular cylinders. It may be laid down as an absolute rule that for an engineering structure to be safe, there should never occur any discontinuity in the shearing stress across any surface inside the material or on its boundary. It is true that in most cases the stress will be relieved by plastic flow and the variation of the shear will become continuous, though rapid. But such points, especially the point *from which* the shear starts acting $(-a', 0)$, where the infinite stress is a *tension*, will remain points of weakness and danger.

§ 37. *Application of Solutions of § 33 to the Case of Tension Produced by Shearing Stress Applied to the Edges.*

In practice test-pieces for tension are usually strained by pressure applied to projecting collars, the latter transmitting this pressure to the body of the material in the shape of shear. In no case can we apply tension directly to the ends of a bar. It is therefore important to know how far the effect of the method of application of the total pull disturbs the usual solution for a uniform tension.

Let us then consider the effect of having two concentrated shears L, one as before acting at the point $(0, b)$, and another equal and parallel to the first acting at the point $(0, -b)$. By superimposing on one another two solutions of the type obtained in § 33, we get the solution required. It will be found that this solution gives a tension $L/2b$ over the left-hand extremity of the beam and a pressure $L/2b$ over the right-hand end.

If we require to have no tension over the right-hand end, and a uniform tension L/b at the left-hand end, to balance the shears, we have to introduce the uniform tension solution $P = L/2b$, $Q = 0$, $S = 0$, $U = Lx/2bE$, $V = \eta Ly/2bE$; we eventually find

$$U = -\frac{\lambda' + 2\mu}{16\mu(\lambda' + \mu)}\frac{Lx^2}{ab} + \frac{3\lambda' + 2\mu}{16\mu(\lambda' + \mu)}\frac{Ly^2}{ab} + \frac{Lx}{2bE}$$

$$+ \sum_1^\infty \frac{L}{ma}\frac{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\cosh mb - \frac{mb}{\mu}\sinh mb}{\sinh 2mb + 2mb}(\cosh my \cos mx - 1)$$

$$+ \sum_1^\infty \frac{L}{m\mu a}\frac{(my \cosh mb)}{\sinh 2mb + 2mb}\sinh my \cos mx$$

$$V = \frac{\lambda'}{8\mu(\lambda' + \mu)}\frac{Lxy}{ab} + \frac{L\eta y}{2bE} + \sum_1^\infty \frac{L}{m\mu a}\frac{\cosh mb}{\sinh 2mb + 2mb} my \cosh my \sin mx$$

$$- \sum_1^\infty \frac{L}{ma}\frac{\frac{1}{\lambda' + \mu}\cosh mb + \frac{1}{\mu} mb \sinh mb}{\sinh 2mb + 2mb}\sinh my \sin mx$$

$$P = \frac{L}{2ab}(a - x) - \sum_1^\infty \frac{L}{a}\frac{4\cosh mb - 2mb \sinh mb}{\sinh 2mb + 2mb}\cosh my \sin mx$$

$$- \sum_1^\infty \frac{2L}{a}\frac{\cosh mb}{\sinh 2mb + 2mb} my \sinh my \sin mx$$

$$Q = -\sum_1^\infty \frac{L}{a}\frac{2mb \sinh mb}{\sinh 2mb + 2mb}\cosh my \sin mx$$

$$+ \sum_1^\infty \frac{2L}{a}\frac{\cosh mb}{\sinh 2mb + 2mb} my \sinh my \sin mx$$

$$S = \frac{Ly}{2ab} + \sum_1^\infty \frac{2L}{a}\frac{\cosh mb}{\sinh 2mb + 2mb} my \cosh my \cos mx$$

$$+ \sum_1^\infty \frac{2L}{a}\frac{(\cosh mb - mb \sinh mb)}{\sinh 2mb + 2mb}\sinh my \cos mx.$$

If a be made to tend towards infinity, we get the expressions:

$$\left.\begin{aligned}
U &= \frac{Lx}{2bE} + \frac{L}{\pi}\int_0^\infty \frac{1}{u}\,\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u - \frac{1}{\mu}u\sinh u}{\sinh 2u + 2u}\left(\cosh\frac{uy}{b}\cos\frac{ux}{b} - 1\right)du \\
&\quad + \frac{Ly}{\pi b}\int_0^\infty \frac{1}{\mu}\,\frac{\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
V &= \frac{L\eta y}{2bE} - \frac{L}{\pi}\int_0^\infty \frac{1}{u}\,\frac{\frac{1}{\lambda'+\mu}\cosh u + \frac{1}{\mu}u\sinh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&\quad + \frac{Ly}{\pi b}\int_0^\infty \frac{1}{\mu}\,\frac{\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
P &= \frac{L}{2b} - \frac{2L}{\pi b}\int_0^\infty \frac{2\cosh u - u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&\quad - \frac{2Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
Q &= -\frac{2L}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&\quad + \frac{2Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
S &= \frac{2Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&\quad + \frac{2L}{\pi b}\int_0^\infty \frac{\cosh u - u\sinh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du
\end{aligned}\right\} \quad (108).$$

§ 38. *Correction to be Applied in this Case to the Stretch along the Edges as we approach the Points of Application of the Load.*

One of the most interesting points about a problem of this kind is to find out at what distance from the region of loading the stretch parallel to the axis takes the value it should have on the uniform tension hypothesis. In practice all measurements of Young's modulus for bars are made by observing the stretch between two points marked on the outer surface of the bar. It is of importance to know the error introduced as we bring these points closer to the places where the stress is applied.

Let us therefore see how the stretch dU/dx varies as we go away from the points of application of the load, keeping upon either edge of the beam. If we differentiate the expression (108) for U with regard to x and then write $y = b - y'$ and transform the expression as in § 16, we get easily

$$\frac{dU}{dx} = \frac{L}{2bE} - \frac{2L}{\pi E}\frac{x}{r'^2} + \frac{L}{\pi\mu}\frac{xy'^2}{r'^4} - \frac{2L}{E\pi b}\int_0^\infty \frac{1-2u+e^{-2u}}{\sinh 2u + 2u}\cosh\frac{uy'}{b}\sin\frac{ux}{b}\,du$$
$$- \frac{L}{(\lambda'+\mu)\pi b}\int_0^\infty \frac{u}{\sinh 2u + 2u}\sinh\frac{uy'}{b}\sin\frac{ux}{b}\,du - \frac{Ly'}{\pi\mu b^2}\int_0^\infty \frac{u^2}{\sinh 2u+2u}\cosh\frac{uy'}{b}\sin\frac{ux}{b}\,du$$
$$- \frac{Ly'}{2\pi\mu b^2}\int_0^\infty \frac{1-2u+e^{-2u}}{\sinh 2u+2u}\sinh\frac{uy'}{b}\sin\frac{ux}{b}\,du.$$

Putting in this $y' = 0$,

$$\left(\frac{dU}{dx}\right)_{y'=0} = \frac{L}{2bE} - \frac{2L}{\pi E}\frac{1}{x} - \frac{2L}{\pi bE}\int_0^\infty \frac{1-2u+e^{-2u}}{\sinh 2u+2u}\sin\frac{ux}{b}\,du.$$

Now the last integral may be written

$$\int_0^\infty \left(\frac{1-2u+e^{-2u}}{\sinh 2u+2u} + e^{-\frac{uh}{b}} - \frac{1}{2u}\right)\sin\frac{ux}{b}\,du$$
$$+ \frac{\pi}{4} - \frac{bx}{h^2+x^2} \text{ if } x \text{ is positive,}$$

and if x is negative, then $-\pi/4$ must be written instead of $+\pi/4$. h is any positive constant.

Now the function

$$f(u) = \frac{1-2u+e^{-2u}}{\sinh 2u + 2u} + e^{-\frac{uh}{b}} - \frac{1}{2u}$$

is such that $f(0) = f(\infty) = 0$, $f'(\infty) = 0$ and $\int_0^\infty | f'(u) |\, du$ is finite. It follows therefore from reasoning similar to that given on p. 107 that $\int_0^\infty f(u)\sin\frac{ux}{b}\,du$ tends to zero as x increases.

Hence, if x be positively increasing $\left(\frac{dU}{dx}\right)_{y'=0}$ tends to 0, and if x be negatively increasing $\left(\frac{dU}{dx}\right)_{y'=0}$ tends to L/bE, as it should.

The values of the integral, calculated for various values of the ratio x/b, have given the following values for $\left(\frac{dU}{dx}\right)_{y'=0}$, as compared with its value for a uniform tension L/b.

x/b	$\left(\frac{dU}{dx}\right)_{y'=0} \Big/ \left(\frac{L}{bE}\right)$
$-\pi$	·997
$-2\pi/3$	·982
$-\pi/2$	·985
$-\pi/3$	1·084
$-\pi/6$	1·652
$+\pi/6$	−·652
$+\pi\ 3$	−·084
$+\pi/2$	·015
$+2\pi/3$	·018
$+\pi$	·003

We see theretore that the stretch reaches its limiting value with very great rapidity. At a distance from the point of application of the load equal to about $1\frac{1}{2}$ times the greatest breadth $2b$ of the bar the error in the stretch is only 3/1000. In fact the stretch begins to get near its limiting value at a much earlier stage than this, the error being less than 10 per cent. at a distance from the load of about half the greatest breadth.

We find therefore that in this case also the *distribution* of the load becomes practically indifferent as soon as we come to distances from the load which are of the same order of magnitude as the greatest dimension of the cross-section. As a practical rule, when accurate measurements are to be taken, it will be advisable to keep always a length varying from 1 to $1\frac{1}{2}$ times this greatest dimension between the points where the stress-system is applied and those at which measurements are taken.

PART V.

Solutions in Finite Terms; Special Application to the Case of a Beam Carrying a Uniform Load.

§ 39. *Solutions in Finite Terms.*

If in (21)–(25) of pp. 70, 71 we write

$$\left.\begin{aligned} \phi(\xi) &= \tfrac{1}{2}\frac{\lambda' + 2\mu}{\lambda' + \mu}(A_n + iB_n)\,\xi^n \\ \chi(\eta) &= \tfrac{1}{2}\frac{\lambda' + 2\mu}{\lambda' + \mu}(A_n - iB_n)\,\eta^n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

$$\left.\begin{aligned} G_1(\xi) &= \frac{1}{2\mu}(C_n + iD_n)\,\xi^n \\ F_1(\eta) &= \frac{1}{2\mu}(C_n - iD_n)\eta^n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (109),$$

we obtain the following homogeneous solutions in x, y.

$$\left.\begin{aligned} U &= \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}(A_n u_n + B_n v_n) - \frac{ny}{4\mu}(A_n v_{n-1} - B_n u_{n-1}) + \frac{1}{2\mu}(C_n u_n + D_n v_n) \\ V &= \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}(A_n v_n - B_n u_n) - \frac{ny}{4\mu}(A_n u_{n-1} + B_n v_{n-1}) - \frac{1}{2\mu}(C_n v_n - D_n u_n) \\ P &= A_n\left(\frac{3n}{4}u_{n-1} - \frac{n(n-1)}{2}\,y v_{n-2}\right) + B_n\left(\frac{3n}{4}v_{n-1} + \frac{n(n-1)}{2}\,y u_{n-2}\right) \\ &\qquad + n(C_n u_{n-1} + D_n v_{n-1}) \\ Q &= A_n\left(\frac{n}{4}u_{n-1} + \frac{n(n-1)}{2}\,y v_{n-2}\right) + B_n\left(\frac{n}{4}v_{n-1} - \frac{n(n-1)}{2}\,y u_{n-2}\right) \\ &\qquad - n(C_n u_{n-1} + D_n v_{n-1}) \\ S &= A_n\left(-\frac{n}{4}v_{n-1} - \frac{n(n-1)}{2}\,y u_{n-2}\right) + B_n\left(\frac{n}{4}u_{n-1} - \frac{n(n-1)}{2}\,y v_{n-2}\right) \\ &\qquad - n(C_n v_{n-1} - D_n u_{n-1}) \end{aligned}\right\} . \ (110),$$

where u_n, v_n are the two homogeneous solutions of $\dfrac{d^2\phi}{dx^2} + \dfrac{d^2\phi}{dy^2} = 0$, thus,

$$u_n = x^n - \frac{n(n-1)}{1\,.\,2}\,x^{n-2}y^2 + \ldots$$

$$v_n = nx^{n-1}y - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,x^{n-3}y^3 + \ldots$$

and $u_0 = 1$, $v_0 = 0$, $u_{-1} = 0$, $v_{-1} = 0$.

We may add any number of such polynomial solutions. If we take n of them, beginning with $n = 1$, and in the expressions (110) write $y = \pm b$, we find $(Q)_{+b}$, $(Q)_{-b}$, $(S)_{+b}$ and $(S)_{-b}$ each equal to algebraic polynomials in x of degree $(n - 1)$.

Also, since A_1, B_1, C_1, D_1 come in only in the form $\dfrac{A_1}{4} - C_1$, $\dfrac{B_1}{4} + D_1$, they are equivalent to only *two* constants. We have therefore $(4n - 2)$ constants free.

Now these are not enough to make Q and S coincide with any two given polynomials on the upper and lower faces of the beam. Obviously, however, the term containing x^{n-1} both in Q and S is independent of y and therefore cannot satisfy a perfectly general condition. If we make this term disappear by writing $C_n = A_n/4$, $D_n = -B_n/4$, we have now only $4n - 4$ free constants left, but our polynomials being now of the $(n-2)^{\text{th}}$ degree, we should have enough constants to be able to identify Q and S with any two given polynomials on either face of the beam.

As a matter of fact this is not so; for there are solutions, namely, those for: (i.) a uniform longitudinal tension, (ii.) a pure bending couple, (iii.) bending with constant shear, which make Q and S zero over both faces and yet do not annul all the $4n-4$ free constants. There must therefore be relations between the $4n-4$ equations giving the constants. They are not all independent and, consequently, not every system of surface stress expressible in polynomials corresponds to a solution of this type.

§ 40. *Case of* $n = 4$.

Let us see what surface conditions can be satisfied by the solutions of the fourth order.

In this case, remembering $D_4 = -B_4/4$,

$$\begin{aligned} Q = &\left(\frac{A_1}{4} - C_1\right) + \left(\frac{A_2}{2} - 2C_2\right)x + \left(-\frac{B_2}{2} - 2D_2\right)y \\ &+ \left(\frac{3A_3}{4} - 3C_3\right)x^2 + \left(-\tfrac{3}{2}B_3 - 6D_3\right)xy + \left(\tfrac{9}{4}A_3 + 3C_3\right)y^2 \\ &+ 12A_4xy^2 + (-3B_4 - 12D_4)x^2y + (5B_4 + 4D_4)y^3 \end{aligned}$$

$$\begin{aligned} S = &\left(\frac{B_1}{4} + D_1\right) + \left(\frac{B_2}{2} + 2D_2\right)x + \left(-\frac{3A_2}{2} - 2C_2\right)y \\ &+ \left(\frac{3B_3}{4} + 3D_3\right)x^2 + \left(-\tfrac{9}{2}A_3 - 6C_3\right)xy + \left(-\tfrac{15}{4}B_3 - 3D_3\right)y^2 \\ &+ (-9A_4 - 12C_4)x^2y - 12B_4xy^2 + (7A_4 + 4C_4)y^3, \end{aligned}$$

and

$$\begin{aligned} U = &\frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}\left\{\begin{array}{l} A_0 + A_1x + A_2(x^2 - y^2) + A_3(x^3 - 3xy^2) + A_4(x^4 - 6x^2y^2 + y^4) \\ \quad + B_1y + 2B_2xy + B_3(3x^2y - y^3) + B_4(4x^3y - 4xy^3) \end{array}\right\} \\ &+ \frac{1}{4\mu}\left\{\begin{array}{l} B_1y + 2B_2xy + 3B_3(x^2y - y^3) + 4B_4(x^3y - 3xy^3) \\ \quad - 2A_2y^2 - 6A_3xy^2 - 4A_4(3x^2y^2 - y^4) \end{array}\right\} \\ &+ \frac{1}{2\mu}\left\{\begin{array}{l} C_0 + C_1x + C_2(x^2 - y^2) + C_3(x^3 - 3xy^2) + C_4(x^4 - 6x^2y^2 + y^4) \\ \quad + D_1y + 2D_2xy + D_3(3x^2y - y^3) + D_4(4x^3y - 4xy^3) \end{array}\right\} \end{aligned}$$

$$\begin{aligned} V = &\frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}\left\{\begin{array}{l} -B_0 - B_1x - B_2(x^2 - y^2) - B_3(x^3 - 3xy^2) - B_4(x^4 - 6x^2y^2 + y^4) \\ \quad + A_1y + 2A_2xy + A_3(3x^2y - y^3) + A_4(4x^3y - 4xy^3) \end{array}\right\} \\ &- \frac{1}{4\mu}\left\{\begin{array}{l} A_1y + 2A_2xy + 3A_3(x^2y - y^3) + 4A_4(x^3y - 3xy^3) \\ \quad + 2B_2y^2 + 6B_3xy^2 + 4B_4(3x^2y^2 - y^4) \end{array}\right\} \\ &+ \frac{1}{2\mu}\left\{\begin{array}{l} D_0 + D_1x + D_2(x^2 - y^2) + D_3(x^3 - 3xy^2) + D_4(x^4 - 6x^2y^2 + y^4) \\ \quad - C_1y - 2C_2xy - C_3(3x^2y - y^3) - C_4(4x^3y - 4xy^3) \end{array}\right\} \end{aligned}$$

$$\begin{aligned}P = &\left(\frac{3A_1}{4} + C_1\right) + x\left(\frac{3A_2}{2} + 2C_2\right) + y\left(\frac{5B_2}{2} + 2D_2\right)\\ &+ x^2\left(\frac{9A_3}{4} + 3C_3\right) + xy\left(\tfrac{15}{2}B_3 + 6D_3\right) + y^2\left(-\tfrac{21}{4}A_3 - 3C_3\right)\\ &+ x^3\left(3A_4 + 4C_4\right) + x^2y\left(15B_4 + 12D_4\right) + xy^2\left(-21A_4 - 12C_4\right)\\ &+ y^3\left(-9B_4 - 4D_4\right),\end{aligned}$$

and we notice that, in virtue of the relation $B_4 = -4D_4$ the coefficient of x^2y in Q goes out. Hence the coefficient of x^2 is the same for Q_{+b} and Q_{-b}. This alone shows that the solution is not the most general that can be got, given that the stresses on the upper and lower surfaces are quadratic functions of x.

§ 41. *Determination of the Constants for a Beam Uniformly Loaded.*

Here we have, over the upper surface $y = +b$: $Q = \text{constant} = q$ say; over $y = -b$: $Q = 0$; and over $y = \pm b$: $S = 0$.

The last two conditions imply

$$B_1 + 4D_1 + b^2\left(-\frac{15B_3}{4} - 3D_3\right) = 0 \quad \text{(111).}$$

$$\frac{B_2}{2} + 2D_2 - 12B_4b^2 = 0 \quad \text{(112).}$$

$$-\frac{3A_2}{2} - 2C_2 + b^2\left(7A_4 + 4C_4\right) = 0 \quad \text{(113).}$$

$$-\tfrac{9}{2}A_3 - 6C_3 = 0 \quad \text{(114).}$$

$$\frac{3B_3}{4} + 3D_3 = 0 \quad \text{(115).}$$

$$-9A_4 - 12C_4 = 0 \quad \text{(116).}$$

(116) and

$$4C_4 = A_4 \quad \text{(117),}$$

give at once

$$A_4 = 0, \quad C_4 = 0 \quad \text{(118).}$$

The conditions for Q give

$$\frac{A_1}{4} - C_1 - \left(\frac{B_2}{2} + 2D_2\right)b + b^2\left(\frac{9A_3}{4} + 3C_3\right) + b^3\left(5B_4 + 4D_4\right) = q \quad \text{(119).}$$

$$\frac{A_1}{4} - C_1 + \left(\frac{B_2}{2} + 2D_2\right)b + b^2\left(\frac{9A_3}{4} + 3C_3\right) - b^3\left(5B_4 + 4D_4\right) = 0 \quad \text{(120).}$$

$$-\tfrac{3}{2}B_3 - 6D_3 = 0 \quad \text{(121).}$$

$$\frac{A_2}{2} - 2C_2 + b^2\left(12A_4\right) = 0 \quad \text{(122).}$$

$$\frac{3A_3}{4} - 3C_3 = 0 \quad \text{(123).}$$

$$-3B_4 - 12D_4 = 0 \quad \ldots \ldots \ldots \ldots \quad (124).$$

(124) is identically satisfied since $4D_4 = -B_4$.
Also (122), (118), (113) imply

$$A_2 = C_2 = 0 \quad \ldots \ldots \ldots \ldots \quad (125).$$

(119), (120), (112) lead to

$$\left.\begin{aligned} B_2 + 4D_2 &= -\frac{3q}{2b} \\ B_4 &= -\frac{q}{16b^3} \\ A_1 - 4C_1 &= 2q \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (126).$$

Also (123), (114) imply

$$A_3 = C_3 = 0 \quad \ldots \ldots \ldots \ldots \quad (127).$$

(121), (115) are identical. Together with (111) they give

$$\left.\begin{aligned} B_3 &= \frac{1}{3b^2}(B_1 + 4D_1) \\ D_3 &= -\frac{1}{12b^2}(B_1 + 4D_1) \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (128).$$

Equations (118), (124), (125), (126), (127), (128) contain the solution we require. If we substitute the values of the constants in the expressions for the displacements and stresses, we find, after some reductions :

$$\begin{aligned} U = \text{const.} &+ x\left\{A_1 \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)} + \frac{C_1}{2\mu}\right\} + \frac{B_1 y}{E} + \frac{B_2}{E} 2xy \\ &+ \left(\frac{B_1}{4} + D_1\right)\left\{\frac{y}{2\mu} + \frac{4}{b^2}\left(\frac{x^2 y}{E} - \frac{y^3}{3}\left(\frac{1}{E} + \frac{1}{2\mu}\right)\right)\right\} \\ &- \frac{q}{4b^3}\left\{\frac{x^3 y}{E} - xy^3\left(\frac{1}{E} + \frac{1}{2\mu}\right)\right\}, \end{aligned}$$

$$\begin{aligned} V = \text{const.} &- y\left\{A_1 \frac{\lambda' - \mu}{8\mu(\lambda' + \mu)} + \frac{C_1}{2\mu}\right\} - \frac{B_1}{E} x - \frac{B_2}{E}(x^2 - \eta y^2) \\ &+ \left(\frac{B_1}{4} + D_1\right)\left\{\frac{4x^3}{3Eb^2} - \eta \frac{4xy^2}{Eb^2}\right\} + \frac{q}{16b^3}\left\{\frac{x^4}{E} - \eta \frac{6x^2 y^2}{E} + y^4\left(\frac{1}{E} - \frac{1}{\mu}\right)\right\}, \end{aligned}$$

$$\begin{aligned} P = \left(\frac{3A_1}{4} + C_1\right) &+ 2B_2 y - \frac{3qy}{2b} + \frac{3xy}{b^2}\left(\frac{B_1}{4} + D_1\right) \\ &- \frac{3}{4}\frac{x^2 yq}{b^3} + \frac{qy^3}{2b^3}, \end{aligned}$$

$$Q = \frac{q}{2} + \frac{3qy}{4b} - \frac{qy^3}{4b^3},$$

$$S = \left(\frac{B_1}{4} + D_1\right)\left(1 - \frac{y^2}{b^2}\right) - \frac{3}{4}\frac{qx}{b}\left(1 - \frac{y^2}{b^2}\right).$$

In the above terms in A_1, C_1 correspond to a uniform tension along x, the terms B_1 to a rigid body rotation, the terms B_2 to a solution for a pure bending couple, and the terms $\left(\frac{B_1}{4} + D_1\right)$ to a solution for bending under a uniform shear. These various constants can be adjusted according to the conditions at the ends $x = \pm a$.

If, for instance, the total pressure over the ends and the total bending moment are to be zero, the load $2qa$ being balanced by the shear at these ends, we have

$$\frac{3A_1}{4} + C_1 = 0,$$

$$\frac{B_1}{4} + D_1 = 0,$$

$$B_2 = + \tfrac{3}{8}\frac{qa^2}{b^3} + \tfrac{3}{5}\frac{q}{b},$$

and we then have

$$\left.\begin{aligned}
P &= -\tfrac{3}{10}\frac{qy}{b} - \tfrac{3}{4}\frac{x^2yq}{b^3} + \frac{qy^3}{2b^3} + \tfrac{3}{4}\frac{a^2yq}{b^3} \\
Q &= \frac{q}{2} + \frac{3qy}{4b} - \frac{qy^3}{4b^3} \\
S &= -\tfrac{3}{4}\frac{qx}{b}\left(1 - \frac{y^2}{b^2}\right) \\
U &= +\tfrac{3}{4}\frac{qa^2xy}{Eb^3} + \tfrac{6}{5}\frac{qxy}{Eb} - \frac{q}{4b^3}\left\{\frac{x^3y}{E} - xy^3\left(\frac{1}{E} + \frac{1}{2\mu}\right)\right\} \\
V &= -\left(\tfrac{3}{8}\frac{a^2}{b^3} + \frac{3}{5b}\right)\frac{q}{E}(x^2 - \eta y^2) + \frac{q}{16b^3}\left\{\frac{x^4}{E} - \eta\frac{6x^2y^2}{E} + y^4\left(\frac{1}{E} - \frac{1}{\mu}\right)\right\}
\end{aligned}\right\} \quad (129).$$

This is the solution for a beam uniformly loaded on the top over a length $2a$ and held up by shears over its terminal cross-sections. In this way the case which occurred in the general solution, and of which the consideration was postponed in § 9, namely $\alpha_0 \neq \beta_0$, is seen to lead to a fairly simple solution in finite terms.

§ 42. *Remarks on the above Solution.*

The above values (129) for U, V, P, Q, S in the case of a beam carrying a uniform load, lead to the following remarks :—

(1) There is no "Neutral Axis" properly so-called; *i.e.*, although the tension vanishes for $y = 0$, t is not strictly proportional to y, a cubic term being introduced. But if $(a^2 - x^2)/y^2$ be large, which is the case for any beam whose length is large compared with its height, the proportional effect of the terms introduced will be small.

(2) The stress Q is not zero; that is, DE SAINT-VENANT'S assumption, that there is no stress across fibres parallel to the axis of the beam, does not hold. Indeed, it was obvious from the beginning that it would not, seeing that there is a stress Q at the upper surface, by hypothesis.

(3) The distribution of shear at each cross-section is parabolic, and is given in terms of the mean shear by the same formula which holds when the shear is uniform.

(4) $\left(\frac{d^2V}{dx^2}\right)_{y=0} = -\frac{3}{4}\frac{(a^2-x^2)}{Eb^3}q - \frac{6q}{5Eb}.$

The curvature is therefore no longer exactly proportional to the bending moment, but contains an additional constant term. A similar result has been obtained by Professor KARL PEARSON and the author for beams of elliptic cross-section under their own weight ('Quarterly Journal of Mathematics,' vol. 31, p. 90). It has since been shown to hold for beams of all forms of section by Mr. J. H. MICHELL ('Quarterly Journal of Mathematics,' vol. 32).

§ 43. *Historical Summary: Remarks and Criticism.*

It may be of interest to give in this place a short sketch of the previous works on the subject, in so far as they are at present known to me.

LAMÉ, in his 'Leçons sur l'Elasticité' (p. 156 *et seq.*), discusses the general problem of the rectangular block, with the single restriction, that the surface stresses are purely normal and are even functions of the co-ordinates. He fails to determine his constants, except in the particular case where the cubical dilatation throughout the block happens to be previously known. As this condition is never satisfied in any actual problem, the solution is of comparatively little use.

DE SAINT-VENANT, in a classical memoir ('Mémoires des Savants Etrangers de l'Académie des Sciences de Paris,' vol. 14), has given solutions for the rectangular parallelopiped under torsion and flexure. These solutions correspond to the case of terminal stress-systems which are *transmitted* through an otherwise unstressed long bar.

Numerous attempts have been made to solve the problem of the rectangular elastic solid by removing one or more faces to infinity, and thus simplifying the surface conditions.

M. EMILE MATHIEU, in his treatise, 'Théorie de l'Elasticité des Corps Solides,' Paris, 1890 (see also 'Comptes Rendus,' vol. 90, pp. 1272–1274), has given a solution of the problem when it can be reduced to two dimensions. His problem is therefore practically the same as that of this paper, except that he has considered only what I have called case (A) on p. 66, and also, that the length a is not taken to be large and the distribution of stress over the faces $x = \pm a$ is given. The solution is, however, so complex in form, and the determination of the constants, by means of long and exceedingly troublesome series, so laborious, that the results defy all attempts at interpretation.

Dr. CHREE ('Roy. Soc. Proc.,' vol. 44, and 'Roy. Soc. Archives'; also 'Quarterly Journal of Mathematics,' vol. 22) has considered at length the solutions of the equations of elasticity in integral powers of x, y, z, and has applied them to the beam problem. Among other results he has obtained expressions for the terms independent of z of a form similar to (110) of this paper. Incidentally, he verifies a number of DE SAINT-VENANT'S results; but no further application is, I think, made of the two-dimensional terms.

Quite recently, Mr. J. H. MICHELL has investigated the theory of long beams under uniform load ('Quarterly Journal of Mathematics,' vol. 32, pp. 28 *et seq.*). The object appears to be to extend DE SAINT-VENANT'S researches to uniformly loaded beams. Mr. MICHELL deduces several interesting results applicable to beams in general and to the rectangular beam in particular, but, so far as I can see, he makes no claim to having obtained explicitly the complete solution in any case.

The surface conditions, however, may be thinned down still further by removing four faces to infinity, leaving only an infinite plate of finite thickness. The problem in this form has been formally solved by LAMÉ and CLAPEYRON ("Sur l'équilibre intérieur des solides homogènes"; 'Mémoires des Savants Etrangers de l'Académie des Sciences de Paris,' vol. 4, pp. 548–552). Their solution, obtained in the form of quadruple integrals, satisfies the surface stress conditions over the two infinite faces. The objections to this solution are two-fold. In the first place it is difficult of interpretation, and the integrals do not enable us to obtain a clear notion of the separate effects of the various forces applied to the plate. In the second place this solution takes no heed of the conditions at the other four limiting faces of the plate which, we should always remember, although they have been removed to a very large distance away, have not physically disappeared. Given total tensions, shears and couples, applied to the four narrow faces of the plate, will produce stresses that will be transmitted through the plate, exactly as in the case of a bent or twisted bar, and will produce a finite effect at points of the plate infinitely distant from the edges, even though the large plane surfaces should be absolutely free from stress.

In order therefore that LAMÉ and CLAPEYRON'S formulæ may correspond to a physical reality, we must superimpose on their solution another of this transmissional type, such that the total shears and total couples due to the sum of the two solutions are all zero round the contour of the plate. Now the problem of the thick elastic rectangular plate, under given *total* shears and couples round its contour, but otherwise free from stress (which is the analogue for plates of the ordinary tensional and flexural solutions for bars), is another of the unsolved problems of the theory of elasticity and, until it is solved, LAMÉ and CLAPEYRON'S solution, unless it happens of itself to satisfy the conditions of no total force at the edge—which will only be true in special cases—fails.

More recently the same problem has been attacked by M. C. RIBIÈRE in a thesis ("Sur divers Cas de la Flexion des Prismes Rectangles," Bordeaux, 1889; see also 'Comptes Rendus,' vol. 126, pp. 402–404 and 1190–1192) in which he gives a solution in a series of circular and hyperbolic functions. He takes his plate of finite dimensions and built-in (*encastrée*) at the edge. By this term he understands that the edge is constrained to remain plane and vertical, and is subject to no shearing-stress. For other terminal conditions the solution, as M. RIBIÈRE states himself, is insufficient. I find that, if the edges of the plate be removed to infinity, his solutions degenerate into LAMÉ and CLAPEYRON'S integrals, of which they therefore give the true meaning.

M. RIBIÈRE, in the same thesis, has also investigated the two-dimensional case, which has been treated of in the present paper.* I am indebted to M. RIBIÈRE for very kindly communicating to me his thesis, with which I became acquainted after my work had been completed. His solutions are of the form (26) (27) (28), and he determines his coefficients, as far as I can see, by the method used here, but does not transform his expressions further. Like LAMÉ and CHAPEYRON, he restricts his applied surface stresses to be normal and investigates only two special cases.

M. RIBIÈRE takes, as I have done, $m = n\pi/a$. This, by the way, is not absolutely necessary. Another set of solutions might be obtained by taking $m = (2n + 1)\pi/2a$. When a is made very large, as is the case in every one of the problems treated here, either set of solutions will lead to the same final form, provided the total terminal conditions are attended to. M. RIBIÈRE, on the contrary, in order to be able to evaluate his series, which become far more manageable when b/a is large, treats chiefly of cases of thick beams of very short span. Now in this case it is no longer permissible to consider merely the total conditions over the ends $x = \pm a$, and to treat the actual *distribution* over these ends as unimportant. M. RIBIÈRE gets over this difficulty by supposing his beam to be *encastré*, as defined above. The same mathematical condition of fixing is assumed by Professor POCHHAMMER ('CRELLE'S Journal,' vol. 81) when treating in a similar fashion of the beam of circular cross section.

It seems doubtful whether anything of this kind does really occur at an actual built-in end of a beam. Certainly POCHHAMMER and RIBIÈRE'S conditions do not agree with the view taken by DE SAINT-VENANT, who, in his calculation of the deflection for a cantilever, has assumed that the elastic line is not horizontal at the built-in end. In this case, however, LOVE has pointed out that the elastic line may have any small slope at the built-in end, provided we superimpose a suitable rigid body displacement. But both he and DE SAINT-VENANT agree to make the end

* Since writing the above, I find that Professor LAMB ('Proc. Lond. Math. Soc.,' vol. 21, p. 70, paper read December, 1889) has worked out the same problem in the form of a series of circular and hyperbolic functions, but he has left his results in this form, without interpreting them further, and I cannot discover that he has considered end-conditions.

sections distorted. As a matter of fact, what really happens at a built-in end is quite unknown. Under these conditions any solution which makes $U = 0$, $dV/dx = 0$ over the ends must be restricted to the case of an infinite continuous beam resting upon a series of equidistant supports, each at the same vertical height; the load carried by the beam being exactly repeated over each span. A rail under its own weight and carried on sleepers is an approximate example. In this case POCHHAMMER and RIBIÈRE's solutions are *exact*, and it is then legitimate to make the span as small as we please.

In practice such conditions will but rarely occur, because, as is well known, any slight difference in the height of the supports, or in the manner in which the beam bears upon them, will upset the symmetry altogether.

The ultimate step in the process of thinning down the boundary conditions is taken when one of the two boundaries of the infinite plate is itself removed to infinity, leaving only one plane bounding an otherwise unlimited solid.

This problem also has been solved by LAMÉ and CLAPEYRON (*loc. cit.*) in terms of quadruple integrals. In this case the limiting conditions at infinity cease to be important, because, in a solid infinite in three dimensions, finite stresses are not transmitted undiminished from infinity, as in a rod or lamina. The solutions, in fact, will lead to stresses that become zero at infinity. This has been shown by BOUSSINESQ ('Applications des Potentiels à l'Etude de l'Equilibre et du Mouvement des Solides Elastiques,' Paris, GAUTHIER-VILLARS, 1885), who has interpreted LAMÉ and CLAPEYRON's results, and obtained by a new method simple expressions for the stresses in an infinite solid, due to arbitrary surface forces applied to a bounding plane. The same results have been obtained by Professor CERRUTI ("Ricerche intorno all' Equilibrio de Corpi Elastici Isotropi," 'Reale Accademia dei Lincei,' vol. 13, 1881–2) in a different way.

BOUSSINESQ, on p. 280, suggests a possible application of his method to the case of two parallel planes, but he makes no attempt to follow it up.

In two papers in the 'Comptes Rendus' (vol. 94, pp. 1510–1516, and vol. 95, pp. 5–11) he has considered the case when the problem of the infinite plane may be treated as two-dimensional, and there he has tried to extend his method to two parallel planes, but had to fall back upon an assumption mathematically unjustifiable.

§ 44. *Recapitulation of Results and Conclusion.*

Looking back upon the results obtained, we see that the general solution given has enabled us to deal with all the most important statical problems connected with the elastic equilibrium of a long beam, of finite height, in so far as the approximation involved in treating them two-dimensionally is valid; and it will be valid, if the horizontal dimension of the cross-section be either very small or very great.

Incidentally the question of the effect of concentrated loads, whether in the form of pressure or of shear, has been discussed. In the case of a beam doubly supported and carrying a concentrated load in the middle, a convergent series has been obtained, giving the exact correction which the finite height of the beam makes it necessary to apply to BOUSSINESQ'S results for an infinite elastic solid.

The results of this part of the paper have been tested by experiments on glass beams, of which it is hoped to eventually publish an account, and they have been found to agree, on the whole, with observation.

The effects of pressing a block of elastic material which rests on a rigid plane, and the manner in which such pressure is transmitted to the plane have also been investigated. It has been found that the pressure on the plane is limited to a restricted area, outside which the elastic block ceases to be in contact with the plane.

The effects of shearing stress have next been considered, in particular the distortion which it produces in lines parallel to the axis of the beam. As in the case of the circular cylinder and in that of the infinite solid bounded by a plane, shear is found to depress those parts of the material towards which it acts.

It is also found that a discontinuity in the shear applied to the surface—although the shear remains finite—involves one of the other stresses becoming infinite, and so is a source of weakness and danger.

The behaviour of a beam under two concentrated loads, acting in opposite senses upon opposite faces of the beam, has been studied. The manner in which the shear across the section varies as these loads are made to approach each other has been exhibited by various diagrams. They show how rapidly the effects of the particular distribution of any total terminal load die out as we go away from the end. At a distance of the order of the height of the beam, they already begin to be negligible.

At a lesser distance than this, however, such effects may become exceedingly important. The case of rivets is instanced, and it is suggested that the results obtained here may give some information which shall be useful in this connection.

Finally a solution in finite terms is obtained for a beam which carries a uniform load. It is shown that the assumptions of the usual theory of flexure are in this case no longer true, but are approximately true only if the height be very small compared with the span. The correction to the curvature, as calculated from the usual formula, is found to be a constant.

With regard to the numerical work, the arithmetic has been checked wherever possible, and it is believed that no serious error has crept in. The values of the integrals, however, have been obtained by the use of quadrature formulæ, and these may not have given a satisfactory approximation in all cases. The three first decimal places, nevertheless, should be correct. As the numerical work was undertaken chiefly to illustrate fairly large variations and to represent them by diagrams, this accuracy appears sufficient.